MODULES OVER OPERATOR ALGEBRAS,
AND THE MAXIMAL C*−DILATION

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ABSTRACT. We continue our study of the general theory of possibly nonselfadjoint algebras of operators on a Hilbert space, and modules over such algebras, developing a little more technology to connect 'nonselfadjoint operator algebra' with the C*−algebraic framework. More particularly, we make use of the *universal*, or *maximal*, C*−algebra generated by an operator algebra, and C*−*dilations*. This technology is quite general, however it was developed to solve some problems arising in the theory of Morita equivalence of operator algebras, and as a result most of the applications given here (and in a companion paper) are to that subject. Other applications given here are to extension problems for module maps, and characterizations of C*−algebras.

1. Introduction - Modules over operator algebras

In what follows $\mathcal{A}$ is a possibly nonselfadjoint operator algebra, that is, a general norm closed algebra of operators on a Hilbert space. We shall assume that $\mathcal{A}$ has a contractive approximate identity (c.a.i.). Thus any C$^*-$algebra is an operator algebra. The *general theory* of operator algebras, and of representations of, and modules over, such algebras, is lamentably sparse. This is in contrast to the selfadjoint case, namely the C$^*-$algebra theory, and the contrast is easily seen in the lack of certain fundamental tools which are available in the selfadjoint case, such as von Neumann's double commutant theorem. This paper is the latest in a series in which we study the class of all operator algebras and their modules, using the recent perspectives and techniques of 'operator space' theory. One of the basic points of this latter theory (see [16]), is that for many purposes it is not sufficient to study linear spaces of operators between Hilbert spaces in the classical functional analytic framework, namely in terms of norms and bounded linear maps. One must use 'matrix norms' and completely bounded linear maps. We refer the reader to [1, 37, 27], for background on operator spaces and operator algebras and a description of some other work in this area, and to [5] for a leisurely introduction and survey of our work. Our main purpose here is to expose some more connections between 'nonselfadjoint operator algebra' with the C$^*-$algebraic framework. Hitherto many researchers seem to have assumed that there is only one important C$^*-$algebra associated with a nonselfadjoint operator algebra $\mathcal{A}$, namely the C$^*-$envelope of $\mathcal{A}$. In fact there is a lattice of C$^*-$algebras generated by $\mathcal{A}$. The C$^*-$envelope, being the 'smallest', is the easiest to concretely get one's hands on, and has many wonderful properties. However, the maximal C$^*-$algebra C$^*_{max}(\mathcal{A})$ generated by $\mathcal{A}$, which we concentrate on here, has very useful properties which the C*-envelope lacks, and is for some purposes more important.

In this paper we study two kinds of representations of a nonselfadjoint algebra $\mathcal{A}$. The first kind are the completely contractive representations π of $\mathcal{A}$ on a Hilbert space H say. Then H is naturally a left $\mathcal{A}$-module: we shall refer to such a module as a *Hilbert $\mathcal{A}$-module*. Perhaps a better name might be *completely contractive Hilbert $\mathcal{A}$-module*, but we will use the shorter name since we do not care about any other kind here. It is not assumed in this paper that such modules are nondegenerate[1], unless we explicitly say so. If $\mathcal{A}$ is a C$^*-$algebra, it is folklore (but also follows from our Theorem 3.1) that contractive representations on Hilbert space are $*-$representations, and thus automatically completely contractive.

The second type of representation of $\mathcal{A}$, which is more general than the first type, corresponds to what is known as an *operator $\mathcal{A}$-module*, and is explained in more detail below. We shall explore the connections between the study of Hilbert and operator modules over $\mathcal{A}$, and those over $\mathcal{C}$, where $\mathcal{C} = C^*_{max}(\mathcal{A})$ is the maximal, or universal, C$^*-$algebra generated by $\mathcal{A}$. We reserve the symbol $\mathcal{C}$ for this C$^*-$algebra throughout. In §2 we show how to construct $\mathcal{C}$ and give some examples. It turns out, although this is not as obvious as at first glance it appears to be, that the class of operator modules over $\mathcal{C}$, is a subcategory of the class of operator modules over $\mathcal{A}$. We derive this in section 3 from some general (but apparently new) facts about Banach modules over C*-algebras. Moreover every Hilbert or operator $\mathcal{A}$-module 'dilates' to an operator module over $\mathcal{C}$. This process is studied in §3, the central section of this paper, which shows how the category of modules over a nonselfadjoint operator algebra $\mathcal{A}$, and the category of modules over any C$^*-$algebra generated by $\mathcal{A}$, are related. As a first application of this and some related ideas, we give in §4, a characterization

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[1]In this paper, for a (left) Banach module X over $\mathcal{A}$ we assume $\|ax\| \leq \|a\|\|x\|$ for $a \in \mathcal{A}, x \in X$. A left Banach module is said to be *essential* or *nondegenerate* if $\{\sum_{k=1}^{n} a_k x_k : n \in \mathbb{N}, \ a_k \in \mathcal{A}, \ x_k \in X\}$ is dense in X. This is equivalent to saying that for any c.a.i. $\{e_\alpha\}$ in $\mathcal{A}$, $e_\alpha x \to x$ for all $x \in X$. Banach modules are not assumed nondegenerate here unless explicitly stated.

of C$^*-$algebras amongst the operator algebras, in terms of injectivity of certain modules, and in terms of the above dilations.

Of course the main motivation for the machinery developed here, is that certain problems concerning nonselfadjoint algebras should be solvable by transferring them to the selfadjoint framework, and then using C$^*-$algebra techniques. An example of this principle is given in a companion paper [7], where we use all the results developed here in §3, to generalize the main result of [6] to nonselfadjoint operator algebras. This completes the circle of ideas begun in [10] concerning strong Morita equivalence of operator algebras. We devote §5 of the present paper to various other connections between C$^*-$dilations and strong and 'weak' Morita equivalence of operator algebras. The reader may find this a rather complicated application of the dilation, however it was our motivation for developing the technology of the earlier sections. We have no doubt that other, more simple, applications of this technology will follow in the course of time. At present we are working on some connections between these ideas and some interesting problems concerning function algebras [8].

In §5, study of the C$^*-$dilation leads us to define a new notion of Morita equivalence of operator algebras, which we call 'strong subequivalence', which has many of the features one associates with strong Morita equivalence. It is called strong subequivalence because, basically, it is an equivalence which may be dilated to a strong Morita equivalence of the generated C*-algebras. Strong Morita equivalence implies strong subequivalence, but the converse is false. Strong subequivalence is thus a weaker notion than strong Morita equivalence, and thus is easier to check in particular examples, while having many of the same consequences. However, we show that strong Morita equivalence of operator algebras, is the same as strong subequivalence when the last-mentioned dilation is to the *maximal* generated C$^*-$algebras. This may be viewed as a new characterization of strong Morita equivalence of operator algebras. We also show that a subcontext of a C$^*-$algebraic strong Morita equivalence is dilatable if and only if it preserves the C$^*-$algebraic weak Morita equivalence.

In §6 we study a class of operator modules, and C*-modules, which can be associated with any operator space or operator module. We also define, using the maximal C$^*-$dilation, a canonical operator algebra $\mathcal{U}(X)$, which we call the upper linking algebra, associated with any operator bimodule X, which has an appropriate universal property for completely contractive bimodule maps defined on X.

Let us begin by establishing the common symbols and notations. We shall use operator spaces quite extensively, and their connections to operator modules. We refer the reader to [10], [4] and [5] for missing background.

Suppose that π is a completely contractive representation of $\mathcal{A}$ on Hilbert space H, and that X is a closed subspace of $B(H)$ such that $\pi(\mathcal{A})X \subset X$. Then X is a left $\mathcal{A}$-module. We say that such X, considered as an abstract operator space and a left $\mathcal{A}$-module, is a left *operator module* over $\mathcal{A}$. By considering X as an abstract operator space and module, we may forget about the particular H, π used. We shall assume in future, unless we explicitly say to the contrary, that the module action on an operator module X is nondegenerate. It is sometimes useful, and equivalent, to allow X in the definition above, to be a subspace of $B(K, H)$, for a second Hilbert space K. The advantage of this is that it will allow $H = [XK]$ if we wish. (The notation $[YZ]$ in this paper will mean the closure of the linear span of products of terms in Y and Z). An obvious modification of a theorem of Christensen-Effros-Sinclair [15] tells us that the operator modules are exactly the operator spaces which are (nondegenerate) left $\mathcal{A}$-modules, such that the module action satisfies $\|ax\| \leq \|a\|\|x\|$ just as for a Banach module, except that now a and x may be square matrices of the same finite size, with entries in $\mathcal{A}$ and X respectively. In other words, the module action is a 'completely contractive' bilinear map (or equivalently, the module action linearizes to a complete contraction $\mathcal{A} \otimes_h X \to X$, where $\otimes_h$ is the Haagerup tensor product). We write $_{\mathcal{A}}OMOD$ for

the category of left $\mathcal{A}$-operator modules. The morphisms are $_{\mathcal{A}}CB(X, W)$, the *completely bounded* left $\mathcal{A}-$module maps. Unless specified otherwise, when X, W are operator modules or bimodules, when we say '$X \cong W$', or '$X \cong W$ as operator modules', we mean that the implicit isomorphism is a completely isometric module map. If X, W are left $\mathcal{A}$-operator modules then $_{\mathcal{A}}CB(X, W)$ is an operator space, whose operator structure is specified by the natural (algebraic) identification $M_n(_{\mathcal{A}}CB(X, W)) \cong \ _{\mathcal{A}}CB(X, M_n(W))$.

We let $_{\mathcal{A}}HMOD$ be the category of nondegenerate Hilbert $\mathcal{A}$-modules. In [10] we showed how $_{\mathcal{A}}HMOD$ may be viewed as a subcategory of $_{\mathcal{A}}OMOD$ (see the discussion at the end of Chapter 2, and after Proposition 3.8, there). Briefly, if $H \in \ _{\mathcal{A}}HMOD$, then if H is equipped with its Hilbert column operator space structure H^c, then $H^c \in \ _{\mathcal{A}}OMOD$. Conversely, if $V \in \ _{\mathcal{A}}OMOD$ is also a Hilbert column space, then the associated representation $\mathcal{A} \to B(V)$ is completely contractive and nondegenerate. It is well known that for a linear map $T : H \to K$ between Hilbert spaces, the usual norm equals the completely bounded norm of T as a map $H^c \to K^c$. Thus we see that the assignment $H \mapsto H^c$ embeds $_{\mathcal{A}}HMOD$ as a (full) subcategory of $_{\mathcal{A}}OMOD$. In future if a Hilbert space is referred to as an operator space, it will be with respect to its column operator space structure, unless specified to the contrary.

In [5] Lemma 8.1 we showed that if $\mathcal{A}$ is an operator algebra with contractive approximate identity $\{e_\alpha\}$, and if $\mathcal{D}$ is any C$^*-$algebra generated[2] by $\mathcal{A}$, then $\{e_\alpha\}$ is a contractive approximate identity for $\mathcal{D}$. This fact will be used frequently. In particular if follows that the obvious action of $\mathcal{A}$ on $\mathcal{D}$ is nondegenerate, so that $\mathcal{D} \in \ _{\mathcal{A}}OMOD$.

We usually choose work with left modules here. The right module versions, or bimodule versions, are mostly similar. There is an important principle which allows one to go between right and left operator modules. Namely, if V is a left module over $\mathcal{A}$, define $\bar{V} = \{\bar{v} : v \in V\}$, with the conjugate linear structure. Then $\bar{V}$ is a right module over $\mathcal{A}^*$. Of course if $\mathcal{A}$ is a C$^*-$algebra, then $\mathcal{A}^* = \mathcal{A}$, otherwise one can view $\mathcal{A}^*$ as the algebra of adjoints of $\mathcal{A}$ in any containing C$^*-$algebra. There is an obvious operator space structure to put on $\bar{V}$, namely $\|[\bar{v}_{ij}]\|_n = \|[v_{ji}]\|_n$. If V is a left operator module over $\mathcal{A}$ then $\bar{V}$ is a right operator module, and we shall call it the conjugate operator module.

We end this section with a fairly obvious observation:

Lemma 1.1. *Suppose that $\mathcal{D}$ is a C*-algebra generated by $\mathcal{A}$, that H and K are Hilbert $\mathcal{D}$-modules, and that $i : H \to K$ and $q : K \to H$ are contractive $\mathcal{A}-$module maps with $q \circ i = Id_H$. Then i and q are $\mathcal{D}-$module maps. In particular, a unitary $\mathcal{A}-$module map $u : H \to K$ is a $\mathcal{D}-$module map.*

For completeness we give the easy proof. By a basic fact about contractions on a Hilbert space, we have $q = i^*$. Let us write ρ and σ for the representations of $\mathcal{D}$ on H and K respectively. Then for $a \in \mathcal{A}, \zeta \in H, \eta \in K$ we have

$$\langle i(\rho(a)^*\zeta), \eta \rangle = \langle \zeta, \rho(a)q(\eta) \rangle = \langle \zeta, q(\sigma(a)\eta) \rangle = \langle \sigma(a)^* i(\zeta), \eta \rangle .$$

This shows that i is a $\mathcal{D}-$module map. Similarly q is a $\mathcal{D}-$module map.

2. THE MAXIMAL C$^*-$ALGEBRA.

In [13] we defined the universal or maximal C$^*-$algebra of an operator algebra $\mathcal{A}$, and it appeared again in [9]. Since it did not play a particularly significant role in those papers, we did not give a careful development. We begin by remedying this omission.

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[2]That is, $\mathcal{D}$ is a C$^*-$algebra generated by a completely isometric, homomorphic, copy of $\mathcal{A}$.

Definition 2.1. *If $\mathcal{A}$ is an operator algebra with contractive approximate identity, then there exists a C*−algebra $\mathcal{C}$ and a completely isometric homomorphism $i : \mathcal{A} \to \mathcal{C}$ such that $i(\mathcal{A})$ generates $\mathcal{C}$ as a C*−algebra, and such that if $\phi : A \to \mathcal{D}$ is any completely contractive homomorphism into a C*−algebra $\mathcal{D}$, then there exists a (necessarily unique) *-homomorphism $\tilde{\phi} : \mathcal{C} \to \mathcal{D}$ such that $\tilde{\phi} \circ i = \phi$. The C*−algebra $\mathcal{C}$ is called the maximal C*−algebra generated by $\mathcal{A}$, and is written as $C^*_{max}(\mathcal{A})$.*

The existence and uniqueness of such a universal object $(\mathcal{C}, i)$ is not difficult, but since it is not written anywhere in the literature as far as we are aware, we give the details. We may suppose that $\mathcal{A}$ has an identity of norm 1 (otherwise adjoin an identity in the usual way, and let $\mathcal{C}$ be the C*-subalgebra of $C^*_{max}(\mathcal{A}^1)$ generated by $\mathcal{A}$). Let $\mathcal{E}$ be the algebraic free product of $\mathcal{A}$ and $\mathcal{A}^*$, which is clearly a ∗-algebra. We now use some basic facts from [1] or [32] about completely positive maps. We recall that the operator algebra $\mathcal{A}^*$, and indeed the operator system $\mathcal{A} + \mathcal{A}^*$ does not depend on any particular Hilbert space that $\mathcal{A}$ is represented on. Let $\theta : \mathcal{A} \to \mathcal{D}$ be a c.c. homomorphism into a C*−algebra $\mathcal{D}$. Let $\mathcal{D}'$ be the C*−algebra generated by the range of θ. Then θ extends to a completely positive unital map $\mathcal{A} + \mathcal{A}^* \to \mathcal{D}'$, which when restricted to $\mathcal{A}^*$ is a c.c. homomorphism θ^*. Then $\theta \star \theta^* : \mathcal{E} \to \mathcal{D}'$ is a ∗-representation. In the usual way, $\mathcal{E}$ gives rise to a C*−algebra $\mathcal{C}$ by taking the supremum over all such ∗-representations. Clearly $\mathcal{A}$ is unitally completely isometrically embedded as a subalgebra of $\mathcal{C}$, $\mathcal{A}$ generates $\mathcal{C}$, and $\mathcal{C}$ has the required universal property. This gives the existence of $\mathcal{C}$. However, $\mathcal{C}$ is clearly unique in the sense that if $(\mathcal{C}', i')$ is any other pair with the property described in 2.1 then there exists a unique *-isomorphism $\pi : C \to C'$ with $\pi \circ i = i'$.

Proposition 2.2. *We have $C^*_{max}(\mathcal{A}_1 \star \mathcal{A}_2) \cong C^*_{max}(\mathcal{A}_1) \star C^*_{max}(\mathcal{A}_2)$, for operator algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with c.a.i., where $\star$ is the operator algebra free product of [13].*

This follows immediately from the universal properties. The analogous result for the maximal operator algebra tensor product of [34] is certainly false, as may be seen for example from 2.4 below, and 2.6 in [34].

Remark. For those who are familiar with operator space theory, it is tempting to think of $\mathcal{C} = C^*_{max}(\mathcal{A})$ as an infinite Haagerup tensor product of copies of $\mathcal{A}$ and $\mathcal{A}^*$. Indeed it is tempting to think of elements in $\mathcal{C}$, in the spirit of [13], as products $A_1 B_1 A_2 B_2 \cdots$ of matrices, with A_i from $\mathcal{A}$ and B_i from $\mathcal{A}^*$. From this perspective one might be led to conjecture that $\mathcal{C} \otimes_h \mathcal{C} \cong \mathcal{C}$ or $\mathcal{C} \otimes_{h\mathcal{A}} \mathcal{C} \cong \mathcal{C}$. The first conjecture is false unless $\mathcal{A} = \mathbb{C}$ by [2] Theorem 1. The second is also false, as we shall see later in 4.3.

Notice that in 2.1 one may take w.l.o.g. the $\mathcal{D}$ there to be $B(H)$ for a Hilbert space H. That is, we may take the ϕ in 2.1 to be a completely contractive representation. From this we see immediately that Hilbert $\mathcal{A}$-modules are automatically Hilbert $\mathcal{C}$-modules and vice versa. Thus as *objects* $_{\mathcal{A}}HMOD = {}_{\mathcal{C}}HMOD$. However the morphisms in these two categories are not the same, since $\mathcal{A}$-intertwiners are not necessarily $\mathcal{C}$-interwiners. In fact it is clear that $_{\mathcal{C}}HMOD$ is a subcategory of $_{\mathcal{A}}HMOD$.

We remark that it seems interesting to transfer the language of the representation theory of C*−algebras to operator algebras. Thus for example we say that $\mathcal{A}$ (or a representation ϕ of $\mathcal{A}$), is type I or CCR, and so on, if and only if $\mathcal{C}$ (or $\tilde{\phi}$) has this property. For example, by results in [19], the disk algebra is NGCR. This example is discussed further in 2.3.

In the rest of this paper we will take H_u to be the Hilbert space of the universal representation of $\mathcal{C}$, and will refer to H_u as the universal representation of $\mathcal{A}$. It is clear from C*−algebraic representation theory, that any nondegenerate Hilbert $\mathcal{A}$-module is (completely) isometrically $\mathcal{A}$−module isomorphic to a complemented $\mathcal{A}$-submodule of a direct sum of copies of H_u.

Example 2.3.

Consider $\mathcal{A} = A(\mathbb{D})$, the disk algebra. In this case $\mathcal{C} = C^*_{max}(\mathcal{A})$ is the universal C*−algebra generated by a contraction, which has been studied by many researchers. This is a noncommutative C*−algebra, generated by a non-normal contraction z, say.

We found this example helpful in disposing of several incorrect guesses we had concerning C^*_{max}. For example, one can use it to show that if $S \in {}_{\mathcal{A}}CB(\mathcal{C}, \mathcal{C})$, and $S(a) = 0$ for all $a \in \mathcal{A}$, then S is not necessarily the zero map. Define $L(c) = zc$ for $c \in \mathcal{C}$. Clearly $L \in {}_{\mathcal{A}}CB(\mathcal{C})$, but L is not in ${}_{\mathcal{C}}CB(\mathcal{C})$ since z is not normal. If we put $S = L$, notice that S restricted to $\mathcal{A}$ equals r_z, i.e. right multiplication by z. Hence $S - r_z$ is a left $\mathcal{A}$-module map on $\mathcal{C}$, is zero on $\mathcal{A}$, but is not the zero map.

Example 2.4.

Consider $\mathcal{A} = \mathcal{T}(2)$, the upper triangular 2×2 matrices. Let $\mathcal{A}_0$ be its subalgebra consisting of those matrices with repetition on the main diagonal. Then $C^*_{max}(\mathcal{A}_0)$ is the well known universal C*−algebra generated by a nilpotent operator, and $\mathcal{A}_0$ is the universal operator algebra generated by a nilpotent operator. In [24], $C^*_{max}(\mathcal{A}_0)$ is shown to be $\{f \in M_2(C([0,1])) : f(0) \in \mathbb{C}I\}$, also known as the cone over M_2. In fact $C^*_{max}(\mathcal{A}) = \{f \in M_2(C([0,1])) : f(0) \text{ is a diagonal matrix}\}$. We will prove this (and a little bit more). It is convenient to work with the dense subalgebra $\mathcal{E}$ of the last C*−algebra consisting of matrices of the form

$$\begin{bmatrix} b_1 & b_2\sqrt{t} \\ b_3\sqrt{t} & b_4 \end{bmatrix}$$

here 't' is the basic monomial on $[0,1]$, and $b_i \in C([0,1])$. Note that this last C*−algebra is generated by the subalgebra consisting of matrices

$$\begin{bmatrix} \lambda_1 & \mu\sqrt{t} \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1, \lambda_2, \mu \in \mathbb{C}$. This subalgebra is easily seen to be completely isometrically isomorphic to $\mathcal{T}(2)$, and we will henceforth take $\mathcal{A}$ to be this subalgebra. Notice that if $T : K \to H$ is any contractive operator between Hilbert spaces, then the subspace of $B(H \oplus K)$ consisting of matrices

$$\begin{bmatrix} \lambda_1 I & \mu T \\ 0 & \lambda_2 I \end{bmatrix}$$

with $\lambda_1, \lambda_2, \mu$ scalar, is an algebra. The C^*−algebra it generates consists of all the matrices of the form

$$\begin{bmatrix} p_1(TT^*) & p_2(TT^*)T \\ T^*p_3(TT^*) & p_4(T^*T) \end{bmatrix} \, ,$$

where the $p_i \in C([0,1])$. It is easily checked that the map

$$\begin{bmatrix} p_1 & p_2\sqrt{t} \\ p_3\sqrt{t} & p_4 \end{bmatrix} \mapsto \begin{bmatrix} p_1(TT^*) & p_2(TT^*)T \\ T^*p_3(TT^*) & p_4(T^*T) \end{bmatrix} \, ,$$

is a *-homomorphism from $\mathcal{E}$ into $B(H \oplus K)$. However it is clearly continuous - notice for example, that $\|p_2(TT^*)T\| = \|p_2(TT^*)(TT^*)^{\frac{1}{2}}\| \leq \|p_2\sqrt{t}\|_{[0,1]}$. Hence it extends to a *-homomorphism on the containing C*−algebra, and is consequently completely contractive. By restriction we obtain a completely contractive homomorphism from $\mathcal{A}$ into $B(H \oplus K)$. Conversely, any nilpotent operator on a Hilbert space L, or any nondegenerate contractive representation of $\mathcal{A}$, immediately gives a decomposition of L as $H \oplus K$, and an operator T as above, with respect to which we are again in the above situation.

The above shows that $\{f \in M_2(C([0,1])) : f(0)$ is a diagonal matrix$\}$ may be characterized as the universal unital C*−algebra generated by two contractions x, v with relations $x^2 = 0, v^2 = v, vx = x, xv = 0$. This fact is no doubt well known. This seems related to 2.6 in [34] which says, loosely, that a commutant lifting theorem for general operator algebras follows from knowing a certain result for $\mathcal{T}(2)$.

We generalize the previous example in the final section of our paper.

3. Operator modules over a generated C*−algebra and C*−dilations.

This is the central section of this paper, in which we show how a category of modules over a nonselfadjoint operator algebra $\mathcal{A}$, and the category of modules over a C*−algebra generated by $\mathcal{A}$, are related by an interesting pair of adjoint functors. All the results developed here are heavily relied on in [7], and later in the present paper, and should be useful in many other situations.

We begin this section with some general facts about Banach modules over C*−algebras which, as far as we are aware, are new. In [2] we proved the following result:

Theorem 3.1. *Let $\mathcal{D}$ be a C*−algebra, $\mathcal{B}$ a Banach algebra, and $\theta : \mathcal{D} \to \mathcal{B}$ a contractive homomorphism. Then the range of θ is norm-closed, has a contractive approximate identity, and possesses an involution with respect to which it is a C*−algebra.*

Thus if V is a left Banach module over a C*−algebra $\mathcal{D}$, and if we let $\theta : \mathcal{D} \to B(V)$ be the associated contractive homomorphism then the range of θ is a C*−algebra.

Theorem 3.2. *Suppose that V is a Banach module over an operator algebra $\mathcal{A}$ with contractive approximate identity. Write $\theta : \mathcal{A} \to B(V)$ for the associated homomorphism. Suppose that $\mathcal{D}$ is any C*−algebra generated by $\mathcal{A}$. Clearly the $\mathcal{A}$-action on V can be extended to a $\mathcal{D}$-action with respect to which V is a Banach $\mathcal{D}$-module if and only if θ is the restriction to $\mathcal{A}$ of a contractive homomorphism $\phi : \mathcal{D} \to B(V)$. This extended $\mathcal{D}$-action, or equivalently the homomorphism ϕ, is unique if it exists.*

<u>Proof.</u> Only the uniqueness requires proof. We shall require some facts about Banach algebras which may be found in [14]. Suppose that ϕ_1 and ϕ_2 are two contractive homomorphisms $\mathcal{D} \to B(V)$, extending θ. By Theorem 3.1, the ranges $\mathcal{E}_1$ and $\mathcal{E}_2$ of ϕ_1 and ϕ_2 are each C*−algebras, but with possibly different involutions. We will write these involutions as $*$ and $\#$ respectively. With respect to these involutions ϕ_1 and ϕ_2 are '*-homomorphisms'. Choose a c.a.i. $\{e_\alpha\}$ for $\mathcal{A}$, and let $\mathcal{B} = \{T \in B(V) : T\theta(e_\alpha) \to T, \text{ and } \theta(e_\alpha)T \to T\}$. Then $\mathcal{B}$ is a Banach algebra with c.a.i. $\{\theta(e_\alpha)\}$. If $\mathcal{F}$ is a Banach algebra with c.a.i. we define $\mathcal{F}^1 = \mathcal{F}$ if $\mathcal{F}$ is unital, otherwise we let it be the unitization of $\mathcal{F}$, with its 'multiplier norm'. In the nonunital case, it is easy to see that $\mathcal{F}^1$ may be defined equivalently to be the subalgebra of $\mathcal{F}^{**}$ generated by $\mathcal{F}$ and a weak*-limit point of the c.a.i. In any case, the 'unitized' C*−algebras $\mathcal{E}_1^1$ and $\mathcal{E}_2^1$ may be viewed as subalgebras of $\mathcal{B}^1$, with the same unit. Let $a \in \mathcal{A}$, and let f be a state on $\mathcal{B}$ (or equivalently on $\mathcal{B}^1$). Then for $k = 1, 2$, f restricted to $\mathcal{E}_k$ is a state on $\mathcal{E}_k$. Thus $f(\phi_1(a)^*) = \overline{f(\phi_1(a))} = \overline{f(\phi_2(a))} = f(\phi_2(a)^\#)$. Thus

$u = \phi_1(a)^* - \phi_2(a)^\#$ is a Hermitian element in $\mathcal{B}$ (or $\mathcal{B}^1$) with numerical radius 0, and consequently $u = 0$. Therefore $\phi_1 = \phi_2$ on $\mathcal{D}$. $\qquad\qquad\square$

From this we obtain the following 'rigidity' result:

Corollary 3.3. *Let $\mathcal{D}$ be a C^*-algebra generated by an operator algebra $\mathcal{A}$. If V_1 and V_2 are two Banach $\mathcal{D}$-modules, and if $T : V_1 \to V_2$ is an isometric and surjective $\mathcal{A}$-module map, then T is a $\mathcal{D}$-module map.*

Corollary 3.4. *Let $\mathcal{D}$ be a C^*-algebra generated by an operator algebra $\mathcal{A}$. The category of Banach modules over $\mathcal{D}$ is a subcategory of the category of Banach modules over $\mathcal{A}$. Similarly, $_\mathcal{D}OMOD$ is a subcategory of $_\mathcal{A}OMOD$, and $_\mathcal{D}HMOD$ is a subcategory of $_\mathcal{A}HMOD$.*

Thus the 'forgetful functor' from the category of Banach (or operator, or Hilbert) modules over $\mathcal{D}$, to the same category over $\mathcal{A}$, is unambiguous (i.e. one-to-one), and embeds the first category as a subcategory of the second. In more flowery language [25] it turns out that the subcategory is 'reflective'. We regard it as one of the significant open problems in this area to find a good test for when an $\mathcal{A}$-operator module V possesses an extended $\mathcal{C}$-module action.

In the remainder of this section we discuss the '$\mathcal{D}$-dilation' of an $\mathcal{A}$-operator module V, where $\mathcal{D}$ is a C*-algebra generated by $\mathcal{A}$. We shall see that, in the language of category theory, the $\mathcal{D}$-dilation is the left adjoint of the aforementioned forgetful functor from $_\mathcal{D}OMOD$ to $_\mathcal{A}OMOD$. In fact it is a simple 'change of rings'. The word 'dilation', and some of its useful properties, was first introduced in work of Muhly and Na [28, 31], in the case when $\mathcal{D}$ is the C*-envelope of $\mathcal{A}$. We will indicate as we go along, any overlap with their work. The dilation was also used, but not explicitly named as such, in [3] and [9].

Definition 3.5. *A pair (E, i) is said to be a $\mathcal{D}$-dilation of a left $\mathcal{A}$-operator module V, if both of the following hold:*

(*) *E is a left $\mathcal{D}$-operator module and $i : V \to E$ is a completely contractive $\mathcal{A}$-module map,*

(**) *For any left $\mathcal{D}$-operator module V', and any completely bounded $\mathcal{A}-$module map $T : V \to V'$, there exists a unique completely bounded $\mathcal{D}-$module map $\tilde{T} : E \to V'$ such that $\tilde{T} \circ i = T$, and also $\|\tilde{T}\|_{cb} = \|T\|_{cb}$.*

This is a universal property in the sense that if (E', i') are any pair satisfying (*) and (**), then there exists a unique completely isometric $\mathcal{D}$-module isomorphism $\rho : E \to E'$ such that $\rho \circ i = i'$. We will postpone the existence of the $\mathcal{D}$-dilation to the next lemma.

The 'uniqueness' assertion in (**) is equivalent to saying that $i(V)$ generates E as a $\mathcal{D}-$operator module in the obvious sense (namely, that there are no nontrivial closed $\mathcal{D}$-submodules of E which contain V). To see this let $E' = [\mathcal{D}i(V)]^-$, and consider the quotient map $Q : E \to \frac{E}{E'}$.

The $\mathcal{D}$-dilation (E, i) is clearly the unique pair satisfying (*), such that for all $\mathcal{D}$-operator modules V', the canonical map $i^* : {}_\mathcal{D}CB(E, V') \to {}_\mathcal{A}CB(V, V')$, given by composition with i, is an isometric isomorphism. Since $M_n(CB(X, Y)) = CB(X, M_n(Y))$ for operator spaces, it follows that i^* being an isometry for all such V' implies that it is a complete isometry. Thus the $\mathcal{D}$-dilation E of V satisfies:

$$_\mathcal{D}CB(E, V') \cong {}_\mathcal{A}CB(V, V') \ . \qquad\qquad (***)$$

completely isometrically. In the case that $\mathcal{D}$ is the C*-envelope of $\mathcal{A}$, a part of this assertion was observed by Muhly and Na. In fact, what this result says in the language of elementary category theory [25], is that the $\mathcal{D}$-dilation is the left adjoint of the forgetful functor from $_\mathcal{D}OMOD$ to $_\mathcal{A}OMOD$ (discussed at the end of §1). Of course, either of the two compositions of this forgetful functor and the $\mathcal{D}$-dilation is not the identity. Another good name for what we call the $\mathcal{D}-$dilation

might be the '$\mathcal{D}$-adjunct'. In flowery language, this *adjunction* makes $_\mathcal{D}OMOD$ a *reflective sub-category* of $_\mathcal{A}OMOD$.

The following shows that we may take E to be the Haagerup module tensor $\mathcal{D} \otimes_{h\mathcal{A}} V$. See [10] for the definition of the module Haagerup tensor product $\otimes_{h\mathcal{A}}$, as well as for its basic properties, such as the fact that it is associative, functorial, and that $\mathcal{A} \otimes_{h\mathcal{A}} V \cong V$. We note that since $\mathcal{A} \otimes_{h\mathcal{A}} V \cong V$, there is a canonical completely contractive $\mathcal{A}$-module map $i : V \to \mathcal{D} \otimes_{h\mathcal{A}} V$.

Lemma 3.6. *For any left operator module V over $\mathcal{A}$, the $\mathcal{D}$-operator module $E = \mathcal{D} \otimes_{h\mathcal{A}} V$ is the $\mathcal{D}$-dilation of V.*

<u>Proof.</u>   If $T : V \to V'$ is as above, then by the functoriality of the Haagerup tensor product, $Id_\mathcal{D} \otimes T : \mathcal{D} \otimes_{h\mathcal{A}} V \to \mathcal{D} \otimes_{h\mathcal{A}} V'$ is completely bounded. Composing this with the module action $\mathcal{D} \otimes_{h\mathcal{A}} V' \to V'$ gives the required map $\tilde{T}$. Its easy to see that $\tilde{T}$ has the right properties. The uniqueness assertion is obvious.   □

We now make some observations which will be important to us. First, notice that it is not necessary that the V' be nondegenerate in (**) above, since one may always replace V' with its '$\mathcal{D}$-essential submodule'. Note that any T as above maps into this essential submodule of V'. Secondly, observe that by the Christensen-Effros-Sinclair result, it suffices to take $V' = B(H,K)$ in (**), where K is a Hilbert $\mathcal{D}$-module and H is a Hilbert space. The next theorem shows that with a natural qualification, one may as well take $V' = K$.

It was probably first noted by Effros that $_\mathcal{D}CB(F, B(H,K))$ is a dual operator space, if F is a left $\mathcal{D}$-operator module. Using basic results about operator spaces, it can be shown that its operator space predual may be written as $K^r \otimes_{h\mathcal{D}} F \otimes_h H^c$, where K^r is the operator dual of K. The duality pairing is the obvious one, namely, $\langle T, \psi \otimes x \otimes \zeta \rangle = \langle T(x)(\eta), \psi \rangle$, for $T \in {}_\mathcal{D}CB(F, B(H,K)), x \in X, \zeta \in H, \psi \in K^*$. Similarly for $_\mathcal{A}CB(V, B(H,K))$. Note that there is a canonical complete contraction $S : K^r \otimes_{h\mathcal{A}} V \to K^r \otimes_{h\mathcal{D}} E$ formed from the composition of the following maps:

$$K^r \otimes_{h\mathcal{A}} V \overset{Id \otimes i}{\to} K^r \otimes_{h\mathcal{A}} E \cong K^r \otimes_{h\mathcal{D}} \mathcal{D} \otimes_{h\mathcal{A}} E \to K^r \otimes_{h\mathcal{D}} E \ .$$

The last map in this sequence comes from the multiplication $\mathcal{D} \times E \to E$. We then get a map $S_1 = S \otimes Id_H : K^r \otimes_{h\mathcal{A}} V \otimes_h H^c \to K^r \otimes_{h\mathcal{D}} E \otimes_h H^c$. It is easy to check that S_1^* is what we called i^* earlier. Hence i^* is an isometric isomorphism if and only if S_1 is an isometric isomorphism.

If $T : X \to Y$ is a contraction (resp. isometry) between operator spaces, then we will say that T is a *row contraction* (resp. *row isometry*), if $\|[T(x_1) \ T(x_2) \cdots T(X_n)]\| \leq$ (resp. $=$)$\|[x_1 \cdots x_n]\|$ for all n and $x_i \in X$. The following is mostly in [26], but for completeness we give a proof.

Lemma 3.7. *Let $T : X \to Y$ be a linear map between operator spaces. The following are equivalent:*
  (i) T *is a row contraction.*
 (ii) T^* *is a row contraction.*
(iii) *For all Hilbert spaces H, $T \otimes I_H : X \otimes_h H^c \to Y \otimes_h H^c$ is a contraction.*

<u>Proof.</u>   Note that by definition of $\otimes_h$, (i) implies (iii) (and in fact one may replace H^c by any operator space). Put $H = C_n$ in (iii), and observe that $(X \otimes_h C_n)^* \cong CB(X, R_n) \cong R_n(X^*)$. Dualizing $T \otimes I_n$ now yields (ii). Since, therefore, (i) implies (iii), we see that (ii) implies that T^{**} is a row contraction, so that T is also.   □

Putting the observations above together, we obtain the equivalence of (**) and condition (i) or (ii) or (iii) below:

Theorem 3.8. *Suppose a pair (E, i) satisfies (*). Then (E, i) satisfies (**), and consequently is the $\mathcal{D}$-dilation of V, if and only if one of the following properties holds:*

(i) *the canonical map* i^* : $_\mathcal{D}CB(E,K) \to _\mathcal{A}CB(V,K)$ *defined above is a completely isometric isomorphism, for all Hilbert $\mathcal{D}$-modules K.*

(ii) *the canonical map* i^* : $_\mathcal{D}CB(E,K) \to _\mathcal{A}CB(V,K)$ *is a 'row-isometric' isomorphism, for all Hilbert $\mathcal{D}$-modules K.*

(iii) *The canonical map* $S : K^r \otimes_{h\mathcal{A}} V \to K^r \otimes_{h\mathcal{D}} E$ *defined above is a 'row-isometry', for all nondegenerate Hilbert $\mathcal{D}$-modules K.*

It is sufficient in (i) to take K to be the universal representation of $\mathcal{D}$.

   Proof.   Only the last part still requires proof. Every nondegenerate Hilbert $\mathcal{D}$-module K is a complemented submodule of a direct sum of γ copies of the universal representation, where γ is some cardinal. Thus the last assertion of the theorem reduces to proving that: if the restriction map gives a complete isometry $_\mathcal{D}CB(E,H) \cong _\mathcal{A}CB(V,H)$, then also $_\mathcal{D}CB(E,H^\gamma) \cong _\mathcal{A}CB(V,H^\gamma)$ completely isometrically. One way to see this is to first check that $_\mathcal{D}CB(E,H^\gamma) \cong M_{\gamma,1}(_\mathcal{D}CB(E,H))$ (see [17]), and similarly $_\mathcal{A}CB(V,H^\gamma) \cong M_{\gamma,1}(_\mathcal{A}CB(V,H))$. Here $M_{\gamma,1}(X)$, for an operator space X, is the collection of 'columns' of length γ with entries in X, whose truncated finite subcolumns are uniformly bounded.   □

   The last statement of the previous theorem is used in [7].

   If V is an $\mathcal{A}-\mathcal{B}$-operator bimodule, where $\mathcal{B}$ is a second operator algebra with c.a.i., then for any $\mathcal{A}$-operator module V', the space $_\mathcal{A}CB(V,V')$ is naturally a (not necessarily nondegenerate) left $\mathcal{B}$-operator module with respect to the action $(bT)(v) = T(vb)$. We define, as in [10] §2, the space $_\mathcal{A}CB^{ess}(V,V')$ to be the $\mathcal{B}$-essential subspace. This equals $\{T \in _\mathcal{A}CB(V,V') : Tr_{f_\beta} \to T\}$, where $\{f_\beta\}$ is a c.a.i. for $\mathcal{B}$, and r_b is the operation of right multiplication with an element in $\mathcal{B}$. The maps in $_\mathcal{A}CB^{ess}$ we will refer to as '$\mathcal{B}$-essential'. An important motivation for these spaces come from the theory of C*−modules, where the 'imprimitivity C*−algebra' or 'algebra of 'compact' operators', coincides with $_\mathcal{A}CB^{ess}$. We will need the next result later and also in [7]:

Theorem 3.9. *For V an $\mathcal{A}-\mathcal{B}$-operator bimodule, and for any Hilbert space H and any (nondegenerate) Hilbert $\mathcal{A}$-module K, we have that $_\mathcal{A}CB^{ess}(V,B(H,K))$ is weak*-dense in $_\mathcal{A}CB(V,B(H,K))$. Moreover, if $\mathcal{D}$ is a C*-algebra generated by $\mathcal{A}$, and if (E,i) is a $\mathcal{D}-\mathcal{B}$-operator bimodule and a $\mathcal{A}-\mathcal{B}$-module map $i : V \to E$ whose range generates E as a $\mathcal{D}$-operator module, then E is the $\mathcal{D}$−dilation of V if and only if E satisfies (**) for $\mathcal{B}$-essential maps. Moreover, the characterizations of the $\mathcal{D}$-dilation in (i) and (ii) of the previous theorem, remain valid with CB replaced by CB^{ess}.*

   Proof.   If $T \in _\mathcal{A}CB(V,B(H,K))$, then the bounded net $\{f_\beta T\}$ has a weak*-convergent subnet, which easily converges weak* to T. That proves the first assertion. Next notice that if (E,i) satisfy (*), and if i^* is the canonical map $_\mathcal{C}CB(E,B(H,K)) \to _\mathcal{A}CB(V,B(H,K))$ above, then $i^*(T)$ is $\mathcal{B}$-essential if and only if T is $\mathcal{B}$-essential. From this it is easy to see the ' $\Longrightarrow$ ' direction. Conversely, given a complete contraction $T \in _\mathcal{A}CB(V,B(H,K))$, then $f_\beta T$ lifts to a complete contraction S_β in $_\mathcal{C}CB(E,B(H,K))$. A weak*-accumulation point of the S_β will be the desired extension of T. We leave it to the reader to fill in the remaining details.   □

Lemma 3.10. *If V is a left $\mathcal{A}$-operator module, and if $\mathcal{D}$ is a C*−algebra generated by $\mathcal{A}$, then the following are equivalent:*

(i) *there exists a $\mathcal{D}$-operator module V' and a completely isometric $\mathcal{A}$-module map $j : V \to V'$, and*

(ii) *the canonical $\mathcal{A}$-module map $i : V \to \mathcal{D} \otimes_{h\mathcal{A}} V$, is a complete isometry.*

<u>Proof.</u>   Suppose that m is the module action on V'. We have the following sequence of canonical complete contractive $\mathcal{A}$-module maps:

$$V \xrightarrow{i} \mathcal{D} \otimes_{h\mathcal{A}} V \xrightarrow{Id \otimes j} \mathcal{D} \otimes_{h\mathcal{A}} V' \xrightarrow{m} V' \ .$$

These maps compose to j, which yields the assertion.   □

The idea of this last lemma was noticed by Muhly and Na in the case that $\mathcal{D}$ is the C*−envelope $C_e^*(\mathcal{A})$ of $\mathcal{A}$. We will refer to the $C_e^*(\mathcal{A})$-dilation as the 'minimal C*-dilation'. In the case that $\mathcal{D} = \mathcal{C} = C_{max}^*(\mathcal{A})$, we call $\mathcal{C} \otimes_{h\mathcal{A}} V$ the 'maximal C*-dilation'. A major reason for the usefulness of the latter is the following, which follows immediately from the previous result, the Christensen-Effros-Sinclair representation of operator modules, and the fact that every Hilbert $\mathcal{A}$-module is a Hilbert $\mathcal{C}$-module.

Corollary 3.11. *For any left $\mathcal{A}$-operator module V, the canonical $\mathcal{A}$-module map $i : V \to \mathcal{C} \otimes_{h\mathcal{A}} V$, is a complete isometry.*

We will regard V henceforth as an $\mathcal{A}$−submodule of $\mathcal{C} \otimes_{h\mathcal{A}} V$.

There is obviously an analogous C*−dilation for right operator modules, or for operator bimodules. The results in this section carry through without difficulty to these cases.

4. Injectivity and characterizations of C*−algebras.

We now turn to some natural questions about injectivity, C*−dilations, and Hilbert modules which seem to be related. Some of the results in this section may be known to experts, but it seems worthwhile to have them in print.

We will say that a (left) $\mathcal{A}$-operator module Z is (left) $\mathcal{A}$-injective if whenever V_2 is a (left) $\mathcal{A}$-operator module with closed submodule V_1, then every completely bounded $\mathcal{A}$-module map $T : V_1 \to Z$ has a completely bounded $\mathcal{A}$-module map extension $\tilde{T} : V_2 \to Z$, with $\|T\|_{cb} = \|\tilde{T}\|_{cb}$. Other authors do not require this last condition to hold, and perhaps a better name for our property would be *1-injective*. Wittstock showed in [43] that if $\mathcal{D}$ is a unital C*−subalgebra of $B(H)$ then $B(H)$ is $\mathcal{D}$-injective. A rather different proof may be found in [41] (Suen uses bimodules, but the left module case can be easily obtained from his result by standard tricks). The following consequence is fairly trivial, but we don't recall seeing it in the literature. Another possible proof of it, using Suen's method, is described after Theorem 6.1. We reaffirm that we do not assume that Hilbert modules are nondegenerate, unless this is explicitly stated:

Theorem 4.1. *For any Hilbert module H over a C*−algebra $\mathcal{D}$, $B(H)$ is (left) $\mathcal{D}$-injective. More generally, for any other Hilbert space N, $B(N, H)$ is left $\mathcal{D}$-injective, and $B(H, N)$ is right $\mathcal{D}$-injective.*

<u>Proof.</u>   By adjoining I_H to $\mathcal{D}$, Wittstock's result fairly obviously extends to the case when $\mathcal{D}$ is a nonunital C*−subalgebra of $B(H)$ acting nondegenerately on H. Hence if H_u is the universal representation of $\mathcal{D}$, and if K is a direct sum of copies of H_u, then $B(K)$ is $\mathcal{D}$-injective. However, every nondegenerate Hilbert $\mathcal{D}$-module H is a $\mathcal{D}$-complemented submodule of such a K, and if P is the $\mathcal{D}$-module projection onto H, then $PB(K)P \cong B(H)$ as $\mathcal{D}$-operator modules. Thus $B(H)$ is $\mathcal{D}$-injective.

If H is not nondegenerate, we let H' be the essential part of H. To show that $B(H)$ is injective, is sufficient to show that $B(H, H')$ is $\mathcal{D}$-injective, since any $\mathcal{D}$-module map T into $B(H)$ has range inside $B(H, H')$. We may assume H' is nontrivial, otherwise the result is clear. However, by

a routine Hilbert space cardinality argument $B(H, H')$ may be regarded as a $\mathcal{D}$-complemented submodule of $B(K, K)$ where K is a large enough direct sum of copies of H'.

Finally, the $B(N, H)$ case is clear from the above, whereas the right injectivity of $B(H, N)$ follows from the left injectivity of $B(N, H)$ by noting that $B(N, H)$ is the 'conjugate operator module' of $B(H, N)$. $\square$

The connection between injectivity and dilations is explained by:

Proposition 4.2. *Suppose that V_2 is an $\mathcal{A}$-operator module with closed submodule V_1. Suppose that $\mathcal{D}$ is a C^*-algebra generated by $\mathcal{A}$. Then the following are equivalent:*

(i) *The canonical map from the $\mathcal{D}$-dilation of V_1 to $\mathcal{D}$-dilation of V_2 is a complete isometry.*

(ii) *For every $\mathcal{D}$-injective module $\mathcal{B}$, and every completely bounded $\mathcal{A}$-module map $T : V_1 \to \mathcal{B}$, then T has a completely bounded $\mathcal{A}$-module map extension $\tilde{T} : V_2 \to \mathcal{B}$, with $\|T\|_{cb} = \|\tilde{T}\|_{cb}$.*

(iii) *For every Hilbert $\mathcal{D}$-module K, the canonical map $K_r \otimes_{h\mathcal{A}} V_1 \to K_r \otimes_{h\mathcal{A}} V_2$ is a complete isometry, where K_r is the operator dual of K).*

(iv) *Same as (iii), but with a single Hilbert module, namely the Hilbert space of the universal representation of $\mathcal{D}$.*

<u>Proof.</u>   Note that just as in the Remarks after 3.6, it suffices to take $\mathcal{B}$ in (ii) to be $B(H, K)$, where H is a Hilbert space, and K is an Hilbert $\mathcal{D}$-module. By an argument similar to that given in those same Remarks (the main difference being that the map i^* there is a complete quotient map), this is equivalent to (iii) (in fact, one may replace the word 'complete' in (iii) with 'row'). To see that (iv) implies (iii), we first observe that as in 4.1, we may assume K is nondegenerate. Using the functoriality of $\otimes_{h\mathcal{A}}$, and the fact that every nondegenerate Hilbert $\mathcal{D}$-module is a complemented submodule of a direct sum of copies of the universal representation, the result reduces to proving that if (iii) holds for K, then it also holds for K^γ for some cardinal γ. However this is easily seen from the injectivity of the Haagerup tensor product [35, 12], together with the operator space identification $K_r^\gamma \otimes_{h\mathcal{A}} V_k \cong R_\gamma \otimes_h K_r \otimes_{h\mathcal{A}} V_k$, where R_γ is the row Hilbert space of dimension γ. That (i) is equivalent to (ii) follows easily from the universal properties of $\mathcal{D}$-injectivity, and 3.5. For the ' $\implies$ ' direction take a completely contractive $\mathcal{A}$-module map $T : V_1 \to \mathcal{B}$. By 3.6 we get a completely contractive $\mathcal{D}$-module map $\mathcal{D} \otimes_{h\mathcal{A}} V_1 \to \mathcal{B}$. Hence, by our hypothesis and $\mathcal{D}$-injectivity of $\mathcal{B}$, there is a completely contractive $\mathcal{D}$-module map extension $\tilde{T} : \mathcal{D} \otimes_{h\mathcal{A}} V_2 \to \mathcal{B}$. Then $\tilde{T}$ restricted to V_2 is a completely contractive $\mathcal{A}$-module extension of T to V_2. The other direction follows easily by showing that the 'closure of $\mathcal{D} \otimes_{\mathcal{A}} V_1$' in $\mathcal{D} \otimes_{h\mathcal{A}} V_2$ has the correct universal property (in the remark after 3.6). $\square$

Remarks. 1) By symmetry, if we are concerned with right modules, the analogous condition in (iii) would be in terms of spaces $V_k \otimes_{h\mathcal{A}} K$. It is unnecessary to consider the dual space K_r.

2) One may replace the '$\mathcal{D}$' by '$\mathcal{A}$' in condition (iii) and (iv) above, in the case that $\mathcal{D} = \mathcal{C}$.

3) Let us say that a pair (V_1, V_2) satisfying the equivalent conditions of the previous theorem, has the extension property. For example, if there is a completely contractive $\mathcal{A}$-module projection $P : V_2 \to V_1$, then (V_1, V_2) has this property for any such $\mathcal{D}$. In particular if $\mathcal{D} = \mathcal{C}$ and V_2 is a nondegenerate Hilbert $\mathcal{A}$-module with submodule V_1, then (V_1, V_2) has the extension property if and only if the projection of V_2 onto V_1 is an $\mathcal{A}$-module map. Thus if V_1 is a fixed nondegenerate Hilbert $\mathcal{A}$-module, then V_1 is orthogonally injective in the sense of [29] if and only if (V_1, V_2) has the extension property whenever V_2 is a nondegenerate Hilbert $\mathcal{A}$-module containing V_1. We remark that an $\mathcal{A}$-injective Hilbert module is orthogonally injective, fairly clearly. Clearly, 4.2 is related to the topic of 'commutant lifting'.

The following theorem may be viewed as a continuation of the pretty Theorem 3.1 of [29]; where Muhly and Solel give several Hilbert module characterizations of C*−algebras. Indeed the main ingredient of our proof below is the equivalence of (i) and (v) below, which is part of their result. We will therefore not prove this equivalence below.

We found that item (ii) was implied by (vi) or (vii), so that it was natural to conjecture that it alone characterized C*-algebras. After asking him this question, Christian Le Merdy kindly supplied a proof of it using Pisier's δ-norms [23]. Later we found the proof below using Muhly and Solel's result. We will use this fact in the next section.

Theorem 4.3. *The following are equivalent for an operator algebra $\mathcal{A}$ with c.a.i.:*

(i) *$\mathcal{A}$ is a $C^*−$algebra.*

(ii) *For every completely contractive representation $\pi : \mathcal{A} \to B(H)$, the commutant $\pi(\mathcal{A})'$ is selfadjoint.*

(iii) *$B(H)$ is (left) $\mathcal{A}$-injective for every Hilbert $\mathcal{A}$-module H.*

(iv) *Every Hilbert $\mathcal{A}$-module H is $\mathcal{A}$-injective.*

(v) *For every nondegenerate completely contractive representation π of $\mathcal{A}$ on a Hilbert space H, and every $\pi(\mathcal{A})$-invariant closed subspace K of H, $H \ominus K$ is $\pi(\mathcal{A})$-invariant.*

(vi) *$\mathcal{C} \otimes_{h\mathcal{A}} \mathcal{C}$ is completely isometrically isomorphic to $\mathcal{C}$, as a $\mathcal{C} − \mathcal{C}$-operator bimodule.*

(vii) *For every Hilbert $\mathcal{A}$-module H, the dilation $\mathcal{C} \otimes_{h\mathcal{A}} H$ is a Hilbert space.*

(viii) *The canonical map from the $\mathcal{C}$-dilation of V_1 to the $\mathcal{C}$-dilation of V_2 is a complete isometry whenever V_2 is an $\mathcal{A}$-operator module with closed submodule V_1.*

Proof.   By 4.1, (i) implies (iii). Clearly (iii) implies (iv), since H is naturally a complemented $\mathcal{A}$-submodule of $B(H)$. That (iv) implies (v) is in [29], since as we said, an injective Hilbert module is orthogonally injective, but in any case the proof is immediate by extending the inclusion $i : K \subset H$ to a completely contractive $\mathcal{A}$-module map $P \in B(H)$. Clearly P is the projection onto K, and since it is an $\mathcal{A}$-module map we obtain (v). A similar idea shows that (ii) implies (v); if π is as in (v), and if θ is π restricted to K, let $\rho = \theta \oplus \pi$, which is a representation of $\mathcal{A}$ on $K \oplus H$. If i is as above, then

$$T = \left[\begin{array}{cc} 0 & 0 \\ i & 0 \end{array} \right]$$

commutes with ρ. If (ii) holds, T commutes with $\rho(\mathcal{A})^*$, which easily gives (v). Thus (i)-(v) are all equivalent.

Clearly (i) implies (vi). If we have (vi), and if V is any $\mathcal{C}$-operator module, then $\mathcal{C} \otimes_{h\mathcal{A}} V \cong \mathcal{C} \otimes_{h\mathcal{A}} \mathcal{C} \otimes_{h\mathcal{C}} V \cong \mathcal{C} \otimes_{h\mathcal{C}} V \cong V$. Taking V to be a Hilbert $\mathcal{C}$-module shows (vii). Assuming (vii), namely that $K = \mathcal{C} \otimes_{h\mathcal{A}} H$ is a Hilbert space, write i for the canonical map $H \to K$ mentioned in 3.6, and let $T : H \to H$ be the identity map. By 3.5, there is a completely contractive $\mathcal{C}$-module map $\tilde{T} : K \to H$ such that $\tilde{T} \circ i = T$. By Lemma 1.1, i is a $\mathcal{C}−$module map. Hence i is onto, which shows that $\mathcal{C} \otimes_{h\mathcal{A}} H \cong H$. Hence, by the universal property 3.5, given an $\mathcal{A}$-submodule H of any Hilbert $\mathcal{A}$-module K as in (v), the inclusion map $i : H \to K$ is a $\mathcal{C}$-module map; and so we see that (v) holds. Thus (i)-(vii) are all equivalent.

Clearly (i) implies (viii), whereas (viii) implies (iii) by 4.2.   □

Remarks. If H is a nondegenerate Hilbert $\mathcal{A}$-module the proof above shows that $\mathcal{C} \otimes_{h\mathcal{A}} H$ is a Hilbert space if and only if $\mathcal{C} \otimes_{h\mathcal{A}} H \cong H$. As in '(vii) $\implies$ (v)' above, this implies that H is an orthogonally injective module in the sense of [29], and also that the commutant in $B(H)$ of the associated representation of $\mathcal{A}$ on H is selfadjoint. The converse is not true however. Simple calculations show in the case where $\mathcal{A}$ is the disk algebra, then the only Hilbert modules with

$\mathcal{C} \otimes_{h\mathcal{A}} H \cong H$, are one-dimensional. In the case when $\mathcal{A} = \mathcal{T}(2)$, the upper triangular 2×2 matrices, these modules coincide with the Hilbert ℓ_2^∞-modules - in other words the nilpotent part of the action vanishes.

In §5 of [33], Paulsen shows that if H is a Hilbert module over the disk algebra $\mathcal{A} = A(\mathbb{D})$ associated with a coisometry then $B(H)$ is $\mathcal{A}$-injective. Muhly and Solel show in [29] that these H are the 'orthogonally injective Hilbert modules'. This class of modules coincides also with the 1-injective Hilbert $\mathcal{A}$-modules.

Notice that in (viii) one may replace $\mathcal{C}$ by any C$^*-$algebra generated by $\mathcal{A}$.

Finally, one can check that (ii) is equivalent to the universal representation of $\mathcal{A}$ having selfadjoint commutant.

5. MORITA EQUIVALENCE OF OPERATOR ALGEBRAS

In this section $\mathcal{A}$ and $\mathcal{B}$ are operator algebras with c.a.i. We refer the reader to [6, 10] if further background for this section is needed. For the basic theory of Morita equivalence and strong Morita equivalence of C$^*-$algebras we refer the reader to [38, 39, 22].

We begin with a brief discussion of 'weak Morita equivalence'. This is mostly independent of the rest of this section, and the reader could skip to 5.2, if desired. Loosely speaking, this means that two operator algebras have 'the same' Hilbert space representations. More precisely, we say that $\mathcal{A}$ and $\mathcal{B}$ are weakly Morita equivalent if the categories $_{\mathcal{A}}HMOD$ and $_{\mathcal{B}}HMOD$ are naturally isometrically equivalent[3]. It is not hard to show that for C$^*-$algebras $\mathcal{C}$ and $\mathcal{D}$, weak Morita equivalence coincides with what was called 'Morita equivalence'[4] in [38] We note that it is folklore that the latter happens if and only if there is a Hilbert space H such that $e(\mathcal{C})\bar{\otimes}B(H) \cong e(\mathcal{D})\bar{\otimes}B(H)$ *-isomorphically, where $e(\mathcal{C})$ is the enveloping von Neumann algebra of $\mathcal{C}$.

Henceforth we reserve the symbols $\mathcal{C}$ and $\mathcal{D}$ for the maximal C$^*-$algebras generated by $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proposition 5.1. *If $\mathcal{A}$ and $\mathcal{B}$ are weakly Morita equivalent operator algebras then:*
  (i) *If $\mathcal{A}$ is a C$^*-$algebra then so is $\mathcal{B}$.*
  (ii) *$\mathcal{C}$ is weakly Morita equivalent to $\mathcal{D}$.*

<u>Proof.</u>   Suppose that $F : {}_{\mathcal{A}}HMOD \to {}_{\mathcal{B}}HMOD$ is an equivalence functor. For $H \in {}_{\mathcal{A}}HMOD$, we have $_{\mathcal{C}}B(H)$ is a subalgebra of $_{\mathcal{A}}B(H)$. By an obvious argument (see for example Lemma 2.2 in [6]), the map $T \mapsto F(T)$ from $_{\mathcal{A}}B(H)$ to $_{\mathcal{B}}B(H)$ is a isometric homomorphism. Hence its restriction to the C$^*-$algebra $_{\mathcal{C}}B(H)$ is a *-homomorphism, and consequently maps into $_{\mathcal{D}}B(F(H))$.

From this we see that if $\mathcal{A} = \mathcal{C}$, then $_{\mathcal{B}}B(H)$ is a C$^*-$algebra for all Hilbert $\mathcal{B}$-modules. By the implication '(ii) $\implies$ (i)' in Theorem 4.3, we see that $\mathcal{B}$ is a C$^*-$algebra.

Now suppose that $H_1, H_2 \in {}_{\mathcal{A}}HMOD$ and $T \in {}_{\mathcal{C}}B(H_1, H_2)$. Let $H = H_1 \oplus H_2$, and let i_k and q_k be, respectively, the inclusions and projections between the H_k and H. Thus $q_k \circ i_k = Id_{H_k}$, so that $F(q_k)F(i_k) = Id_{F(H_k)}$. From 1.1 it follows that $i_k, q_k, F(q_k)$ and $F(i_k)$ are $\mathcal{C}-$module maps. By the first part, $F(i_2 T q_1)$ is a $\mathcal{C}-$module map. Thus $F(T) = F(q_2)F(i_2 T q_1)F(i_1)$ is a $\mathcal{C}-$module map.

We have shown that F restricts to a functor from $_{\mathcal{C}}HMOD$ to $_{\mathcal{D}}HMOD$. Similarly for G, and now the category equivalence is clear. Note that the natural transformation maps are unitary and

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[3]That is, if there exist contractive functors $F : {}_{\mathcal{A}}HMOD \to {}_{\mathcal{B}}HMOD$ and $G : {}_{\mathcal{B}}HMOD \to {}_{\mathcal{A}}HMOD$, such that $FG \cong Id$ and $GF \cong Id$ naturally isometrically.

[4]In recent years we have heard the term 'weak Morita equivalence' being used for Rieffel's 'Morita equivalence of C$^*-$algebras' (as opposed to his 'strong Morita equivalence').

commute with the action of the operator algebra, and hence also commute with the action of the generated C*−algebra. Thus we have (ii).   □

We refer the reader to [10] for the definition of *strong Morita equivalence* of operator algebras $\mathcal{A}$ and $\mathcal{B}$. Loosely, it is defined in terms of two operator bimodules X and Y, which possess certain pairings $(\cdot) : X \times Y \to \mathcal{A}$ and $[\cdot] : Y \times X \to \mathcal{B}$. The tuple $(\mathcal{A}, \mathcal{B}, X, Y, (\cdot), [\cdot])$ is called a *strong Morita context* (see [10] Definition 3.1). Here we shall usually simply write $(\mathcal{A}, \mathcal{B}, X, Y)$. This generalizes C*−algebraic strong Morita equivalence [39]. If $\mathcal{A}$ and $\mathcal{B}$ are C*−algebras it turns out that X may be taken to be the conjugate bimodule of Y (or equivalently, $X = Y^*$ in the linking C*−algebra [22]).

Definition 5.2.

(i) Suppose that $\mathcal{E}$ and $\mathcal{F}$ are strongly Morita equivalent C*−algebras, and that Z is an $\mathcal{F} - \mathcal{E}$-strong Morita equivalence bimodule, and that $W = \bar{Z}$ is the conjugate $\mathcal{E} - \mathcal{F}$−bimodule of Z. Then we say that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a *C*−Morita context*, or *C*−context* for short.

(ii) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are operator algebras with c.a.i., and suppose that $\mathcal{E}$ and $\mathcal{F}$ are C*−algebras generated by $\mathcal{A}$ and $\mathcal{B}$ respectively. Suppose that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a C*−Morita context, X is a closed $\mathcal{A} - \mathcal{B}$-submodule of W, and that Y is a closed $\mathcal{B} - \mathcal{A}$-submodule of Z. Suppose further that the natural pairings $Z \otimes W \to \mathcal{F}$ and $W \otimes Z \to \mathcal{E}$ restrict to maps $Y \otimes X \to \mathcal{B}$, and $X \otimes Y \to \mathcal{A}$, both with dense range. Then we say that $(\mathcal{A}, \mathcal{B}, X, Y)$ is a *subcontext* of $(\mathcal{E}, \mathcal{F}, W, Z)$. If, further, $\mathcal{E}$ and $\mathcal{F}$ are the maximal C*−algebras of $\mathcal{A}$ and $\mathcal{B}$ respectively, then we shall say that $(\mathcal{A}, \mathcal{B}, X, Y)$ is a *maximal subcontext*. Similarly, a *minimal subcontext* occurs when $\mathcal{E}$ and $\mathcal{F}$ are the C*−envelopes of $\mathcal{A}$ and $\mathcal{B}$.

(iii) A subcontext $(\mathcal{A}, \mathcal{B}, X, Y)$ of a C*−Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$ is said to be *left dilatable* if W is the left $\mathcal{E}$-dilation of X, and Z is the left $\mathcal{F}$-dilation of Y. In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are *left strongly subequivalent*. We also say that X and Y are (left) subequivalence bimodules, and that $(\mathcal{A}, \mathcal{B}, X, Y)$ is a *left subequivalence context*.

There is a similar definition and symmetric theory where we replace the words 'left', by 'right' or 'two-sided'. Generally, we shall omit the word 'two-sided' and simply refer, for example to 'strong subequivalence'.

In order to come to grips with these definitions, we proceed with several observations and examples:

Remarks. Note that (ii) implies that X and Y are nondegenerate operator bimodules over $\mathcal{A}$ and $\mathcal{B}$. This is because W and Z are automatically nondegenerate (see 1.5 in [22]), and any c.a.i for an operator algebra is also a c.a.i. for any C*−algebra it generates.

Write $\mathcal{L}$ for the set of 2×2 matrices

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix}$$

with $a \in \mathcal{A}, b \in \mathcal{B}, x \in X, y \in Y$. Write $\mathcal{L}'$ for the same set, but with entries from the C*−context $(\mathcal{E}, \mathcal{F}, W, Z)$. It is well known (see [22]) that $\mathcal{L}'$ is canonically a C*−algebra, called the 'linking C*−algebra' of Z, or of $(\mathcal{E}, \mathcal{F}, W, Z)$. Saying that $(\mathcal{A}, \mathcal{B}, X, Y)$ is a subcontext of $(\mathcal{E}, \mathcal{F}, W, Z)$ is almost equivalent to saying that $\mathcal{L}$ is a closed subalgebra of $\mathcal{L}'$. We say 'almost', because the latter condition does not imply the statement in (ii) about 'dense range'. In any case it is clear that a subcontext gives a *linking operator algebra* $\mathcal{L}$. Clearly $\mathcal{L}$ has a c.a.i. We shall see that $\mathcal{L}$ generates $\mathcal{L}'$ as a C*−algebra.

If $(\mathcal{A}, \mathcal{B}, X, Y)$ is a subcontext of $(\mathcal{E}, \mathcal{F}, W, Z)$, and if $\mathcal{A}$ and $\mathcal{B}$ are unital, then the pairings in (ii) having dense range is equivalent to (as in Proposition 3.3 of [10]) these pairings being onto,

and hence $(\mathcal{A}, \mathcal{B}, X, Y)$ is a 'c.b.-Morita context' in the sense of [10] Definition 3.1. However, we are mainly interested in when a subcontext is a strong Morita context.

Note that in (iii) we are in the situation where the canonical map from the operator module (X or Y) into its dilation (W or Z) is a complete isometry. We shall see later that there are some simple tests for when a subcontext is left dilatable.

Finally, we remark that we showed in [9] that strong Morita equivalence implies (two-sided) strong subequivalence, and moreover the implicit subcontext may be taken to be minimal (or maximal). Together with K. Jarosz we have found a simple example [8] of a closed subalgebra $\mathcal{A}$ of the disk algebra, giving a strong subequivalence (which is a minimal sucontext, and is of the type discussed in Example 3 below) which is not a strong Morita equivalence. Thus strong subequivalence is genuinely a new notion. We shall see however, that strong Morita equivalence is the same as strong subequivalence via a *maximal* subcontext. Hopefully the distinctions will be illuminated more clearly as we go along. We also refer to [8] for further, and very concrete, illumination of these notions.

Examples 1.) The 'dense range' condition in (ii) is not implied by the dilation condition in (iii). Indeed if $\mathcal{A} = \mathcal{T}(2)$, $\mathcal{B} = \mathbb{C}$, $Y = R_2$, and $X = \mathbb{C}[1 \ 0]^t$, and if $\mathcal{E} = M_2$ (the C*−envelope of $\mathcal{A}$) and $\mathcal{F} = \mathcal{B}$, then it's easily seen that (ii) and (iii) hold with the exception of the pairing $X \otimes Y \to \mathcal{A}$ having dense range. This example is interesting in that in this case the C*−envelopes of $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent with equivalence bimodules being the minimal C*−dilations of X and Y above, but the maximal C*−algebras of $\mathcal{A}$ and $\mathcal{B}$ are not Morita equivalent in any sense.

2.) Another interesting example of subcontexts comes from example 8.2 in [10] (see also 6.9 in [3], where $\mathcal{A} = A(\mathbb{D})$ is the disk algebra, and we find $\mathcal{A}$-operator modules X, Y such that $(\mathcal{A}, \mathcal{A}, X, Y)$ and $(\mathcal{A}, \mathcal{A}, \mathcal{A}, \mathcal{A})$ are two different (two-sided) dilatable subcontexts of $(\mathcal{E}, \mathcal{E}, \mathcal{E}, E)$, where $\mathcal{E} = C(\mathbb{D})$. Hence one cannot hope in general to recover X, Y from the data of $\mathcal{A}, \mathcal{B}$ and the containing C*−context $(\mathcal{E}, \mathcal{F}, W, Z)$. In example 8.3 of [10] we discussed another subcontext coming from matrix algebras of analytic functions, which is not dilatable.

3.) Related to the Example 2, let $\mathcal{A}$ be an operator algebra with identity of norm 1, let $\mathcal{E}$ be a C*−algebra generated by $\mathcal{A}$, and choose $x \in \mathcal{E} \setminus \mathcal{A}$, with x invertible in $\mathcal{E}$, such that $x^{-1}\mathcal{A}x$ generates $\mathcal{E}$ (such as is the case when $x \in \mathcal{A}'$). Then $(\mathcal{A}, x^{-1}\mathcal{A}x, \mathcal{A}x, x^{-1}\mathcal{A})$ is a subcontext of the 'identity context' $(\mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{E})$. One may quite easily write down conditions on x ensuring that this subcontext is left dilatable. For example, suppose that Ω is a compact Hausdorff space, and $\mathcal{A}$ is a uniform algebra on Ω (containing constants and separating points). Let $P = \{|g| : g \in \mathcal{A}\} \subset C(\Omega)_+$. Choose a strictly positive function f on Ω, such that $f, f^{-1} \in P$, or equivalently: $f \in P \cap P^{-1}$. Then it is easy to see from the Stone-Weierstrass theorem that $(\mathcal{A}, \mathcal{A}, \mathcal{A}f, f^{-1}\mathcal{A})$ is a (two-sided) dilatable subcontext of the 'identity context' of $C(\Omega)$. In fact this is true under much less restrictive conditions on f. It appears to be an interesting function algebra question to characterize when such subcontexts are strong Morita contexts (see [8] for more details). For example, it is easy to see that they always are, if $f \in Q$ or $f \in \bar{Q}$ (the uniform closure of Q), where $Q = \{|k| : k, k^{-1} \in \mathcal{A}\} \subset P \subset C(\Omega)$. Note that $P \cap P^{-1} = Q$ if $\mathcal{A}$ is the disk algebra, say, and $\Omega = \mathbb{D}$. This is because if $f = |g|, f^{-1} = |h|, g, h \in \mathcal{A}$, then $|gh| = 1$, and hence by the maximum modulus theorem gh is constant, and hence g is invertible in $\mathcal{A}$.

Proposition 5.3. *If $(\mathcal{A}, \mathcal{B}, X, Y)$ is a subcontext of a C*−Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$, then*

(i) *X and Y generate W and Z respectively as left operator modules. So, for example, W is the smallest closed left $\mathcal{E}$-submodule of W containing X. Similar assertions hold as right operator modules, by symmetry.*

(ii) *The linking operator algebra $\mathcal{L}$ generates the linking C^*-algebra $\mathcal{L}'$ of $(\mathcal{E}, \mathcal{F}, W, Z)$.*

(iii) *If $\mathcal{A}$ or $\mathcal{B}$ is a C^*-algebra, then $(\mathcal{A}, \mathcal{B}, X, Y) = (\mathcal{E}, \mathcal{F}, W, Z)$.*

Proof.   It is easy to see that (ii) and (iii) follow from (i). We shall simply show that X generates W as a left $\mathcal{E}$-operator module. Since the pairing $[\cdot] : Y \otimes X \to \mathcal{B}$ has dense range, we can pick a c.a.i. for $\mathcal{B}$ which is a sum of terms of the form $[y, x]$, for $y \in Y, x \in X$. This c.a.i. is also one for $\mathcal{F}$, and hence sums of terms of the form $w[y, x]$, for $y \in Y, x \in X, w \in W$ are dense in W. However, $w[y, x] = (w, y)x \in \mathcal{E}X$ (where $(\cdot)$ is the other pairing). So X generates W as a left $\mathcal{E}$-operator module.   □

Theorem 5.4. *If $(\mathcal{A}, \mathcal{B}, X, Y)$ is a strong Morita context which is a subcontext of a C^*-Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$, then it is a dilatable subcontext.*

Proof.   By the previous result, X and Y generate W and Z respectively as left operator modules. Thus we have a complete contraction $\mathcal{E} \otimes_{h\mathcal{A}} X \to W$ with dense range. On the other hand

$$W \cong W \otimes_{h\mathcal{B}} \mathcal{B} \cong W \otimes_{h\mathcal{B}} Y \otimes_{h\mathcal{A}} X \cong (W \otimes_{h\mathcal{B}} Y) \otimes_{h\mathcal{A}} X \ .$$

However, the pairing $(\cdot)$ determines a complete contraction $W \otimes_{h\mathcal{B}} Y \to \mathcal{E}$, and so we obtain a complete contraction $W \to \mathcal{E} \otimes_{h\mathcal{A}} X$. One easily checks that the composition of these maps

$$\mathcal{E} \otimes_{h\mathcal{A}} X \to W \to \mathcal{E} \otimes_{h\mathcal{A}} X$$

is the identity, from which it follows that $W \cong \mathcal{E} \otimes_{h\mathcal{A}} X$. Similarly Z is the dilation of Y.   □

Theorem 5.5. *If $(\mathcal{A}, \mathcal{B}, X, Y)$ is a left dilatable maximal subcontext of a C^*-context, then $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent operator algebras, and Y is a strong $\mathcal{A} - \mathcal{B}$-Morita equivalence bimodule, with dual module X. Indeed, it also follows that $(\mathcal{A}, \mathcal{B}, X, Y)$ is a (strong) Morita context. Conversely, every strong Morita equivalence of operator algebras occurs in this way. That is, every strong Morita context is a left dilatable maximal subcontext of a C^*-Morita context.*

Proof.   If $\mathcal{C}$ and $\mathcal{D}$ are as usual the maximal C^*-algebras of $\mathcal{A}$ and $\mathcal{B}$ respectively, and if $(\mathcal{A}, \mathcal{B}, X, Y)$ is a left dilatable subcontext of $(\mathcal{C}, \mathcal{D}, W, Z)$ then, using Lemmas 3.11 and 3.6, we have

$$Y \otimes_{h\mathcal{A}} X \subset \mathcal{D} \otimes_{h\mathcal{B}} (Y \otimes_{h\mathcal{A}} X) \cong Z \otimes_{h\mathcal{A}} X \cong (Z \otimes_{h\mathcal{C}} \mathcal{C}) \otimes_{h\mathcal{A}} X \cong Z \otimes_{h\mathcal{C}} W \cong \mathcal{D} \ ,$$

completely isometrically. On the other hand, we have the canonical complete contraction

$$Y \otimes_{h\mathcal{A}} X \to \mathcal{B} \subset \mathcal{D} \ ,$$

coming from the restricted pairings in (ii). It is easy to check that the composition of the maps in these two sequences agree. Hence the canonical map $Y \otimes_{h\mathcal{A}} X \to \mathcal{B}$ is a completely isometric isomorphism. Similarly, $X \otimes_{h\mathcal{B}} Y \cong \mathcal{A}$ completely isometrically. Thus by the remark before Definition 3.6 in [10] (or see Definition 1.2 in [7] and the 'sketch' beneath it), $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent operator algebras.

The last statement is proved in [9].   □

The last theorem may be viewed as a new characterization of strong Morita equivalence of operator algebras.

We next show that 'strong subequivalence' seems to have many of the nice implications of strong Morita equivalence (see Theorem 4.1 in [10] and the end of Chapter 3 there). We intend to pursue in the near future exactly which other of the consequences of strong Morita equivalence still carry over for this weaker notion. There is presumably also a theory of 'sub-rigged' modules paralleling notions from [3], although we expect to lose some of the rich features of rigged modules.

Theorem 5.6. *Suppose that* $(\mathcal{A}, \mathcal{B}, X, Y)$ *is a left dilatable subcontext of a* $C^*-context$ $(\mathcal{E}, \mathcal{F}, W, Z)$. *Then* $Y \cong {}_{\mathcal{A}}CB^{ess}(X, \mathcal{A})$ *and* $X \cong {}_{\mathcal{B}}CB^{ess}(Y, \mathcal{A})$ *completely isometrically and as operator bimodules, and* $\mathcal{A} \cong {}_{\mathcal{B}}CB^{ess}(Y, Y)$ *and* $\mathcal{B} \cong {}_{\mathcal{A}}CB^{ess}(X, X)$ *completely isometrically and as operator algebras. Moreover, the categories of* $\mathcal{A}$-*submodules of* $\mathcal{E}$-*operator modules, and* $\mathcal{B}$-*submodules of* $\mathcal{F}$-*operator modules, are (completely isometrically) equivalent. This equivalence restricts to an equivalence of the categories of* $\mathcal{A}$-*submodules of Hilbert* $\mathcal{E}$-*modules, and* $\mathcal{B}$-*submodules of Hilbert* $\mathcal{F}$-*modules.*

   Proof.   Write $(\cdot, \cdot)$ and $[\cdot, \cdot]$ for the pairings discussed in (ii) of Definition 5.2. Notice firstly, that there is a natural map $Y \to {}_{\mathcal{A}}CB^{ess}(X, \mathcal{A})$ coming from these pairings. Hence we get a sequence

$$Y \to {}_{\mathcal{A}}CB^{ess}(X, \mathcal{A}) \subset {}_{\mathcal{A}}CB^{ess}(X, \mathcal{E}) \cong {}_{\mathcal{E}}CB^{ess}(W, \mathcal{E}) \cong Z \ ,$$

where the second last map comes from 3.9. However, the composition of maps in this sequence agrees with the inclusion of Y in Z. Hence the map $Y \to {}_{\mathcal{A}}CB^{ess}(X, \mathcal{A})$ is a complete isometry. That this map is onto follows by the argument of [10] Theorem 4.1. A similar proof shows that $X \cong {}_{\mathcal{B}}CB^{ess}(Y, \mathcal{B})$ as operator bimodules, and that $\mathcal{B} \cong {}_{\mathcal{A}}CB^{ess}(X, X)$ and $\mathcal{A} \cong {}_{\mathcal{B}}CB^{ess}(Y, Y)$ (completely isometrically) as operator algebras. Define $F(V) = {}_{\mathcal{A}}CB^{ess}(X, V)$ and $G(U) = {}_{\mathcal{B}}CB^{ess}(Y, U)$, we will show that F and G are are completely contractive equivalence functors between the category operator $\mathcal{A}$-submodules of $\mathcal{E}$-operator modules, and the category of $\mathcal{B}$-submodules of $\mathcal{F}$-operator modules, which compose (up to natural completely isometric isomorphism) to the identity functor.

   If V is an $\mathcal{E}$-operator module, then by 3.9 and [4] Theorem 3.10, we have

$$F(V) = {}_{\mathcal{A}}CB^{ess}(X, V) \cong {}_{\mathcal{E}}CB^{ess}(W, V) \cong Z \otimes_{h\mathcal{E}} V \ .$$

Moreover, this, together with the corresponding result for G, shows that $G(F(V)) \cong W \otimes_{h\mathcal{F}} Z \otimes_{h\mathcal{E}} V \cong \mathcal{E} \otimes_{h\mathcal{E}} V \cong V$.

   For a general $\mathcal{A}$-operator module V, there is a canonical complete contraction $\rho_V : V \to G(F(V)) = {}_{\mathcal{B}}CB^{ess}(Y, {}_{\mathcal{A}}CB^{ess}(X, V))$ given by $(\rho_V(v)(y))(x) = (x, y)v$, for $v \in V, y \in Y, x \in X$. Suppose that V is an $\mathcal{A}$-operator module, and that V' is a $\mathcal{E}$-operator module containing V. Then we get the following sequence of complete contractions

$$V \to G(F(V)) = {}_{\mathcal{B}}CB^{ess}(Y, {}_{\mathcal{A}}CB^{ess}(X, V)) \subset {}_{\mathcal{B}}CB^{ess}(Y, {}_{\mathcal{A}}CB^{ess}(X, V')) \cong V' \ .$$

The first map here is ρ_V. The composition of maps in this sequence is the inclusion map, and so ρ_V is a complete isometry. To show that ρ_V is onto in the unital case is a simple exercise in algebra. In the nonunital case, to show that ρ_V is onto, one may use an argument similar to those in the proof of [10] Theorem 4.1 showing that the maps there are onto. That Hilbert modules in these categories are taken by this equivalence to Hilbert modules follows easily from the observations above.   □

   We recall (see [29] for example) that a Shilov Hilbert module is an $\mathcal{A}$-submodule of a Hilbert module over the C^*-envelope of $\mathcal{A}$. As a consequence it follows that minimal subequivalence of two operator algebras implies a (weak) equivalence between the subcategories of Shilov Hilbert modules. We are led to propose the following definition:

Definition 5.7. *We say that operator algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are (two-sided)* minimally subequivalent *if they are (two-sided) strongly subequivalent, and the* C^*-*algebras in the containing* C^*-*context are the* C^*-*envelopes of* $\mathcal{A}$ *and* $\mathcal{B}$.

   A similar definition pertains where we replace the word 'two-sided' by 'left' or 'right'. Notice that there is no need to define 'maximally subequivalent', since this would coincide with strong

Morita equivalence, by Theorem 5.5. Strong Morita equivalence implies minimal subequivalence by [9]. However, we have examples to show that the converse is false: indeed two-sided minimal subequivalence is a weaker notion than strong Morita equivalence.

We now show how 'strong subequivalence' can arise, by discussing some equivalent conditions for a subcontext $(\mathcal{A}, \mathcal{B}, X, Y)$ of $(\mathcal{E}, \mathcal{F}, W, Z)$ to be left dilatable, or equivalently, for $W \cong \mathcal{E} \otimes_{h\mathcal{A}} X$ and $Z \cong \mathcal{F} \otimes_{h\mathcal{B}} Y$. As we saw in 5.3, the definition of a subcontext already implies that $[\mathcal{E}X] = W$ and $[\mathcal{F}Y] = Z$.

Theorem 3.8 or 3.9 tells us that $W \cong \mathcal{E} \otimes_{h\mathcal{A}} X$ is equivalent to the fact that $_{\mathcal{E}}CB^{ess}(W, H) \cong _{\mathcal{A}}CB^{ess}(X, H)$ completely isometrically for all Hilbert $\mathcal{E}$-modules H. Indeed the Hilbert space of the universal representation would suffice. From C*−module theory (see [4] for background) we have that $_{\mathcal{E}}CB^{ess}(W, H) \cong Z \otimes_{h\mathcal{E}} H$. The last space is a Hilbert column space, whose norm we can completely describe: namely $\|\sum_k z_k \otimes \zeta_k\|^2 = \sum_{k,j}\langle\langle z_k|z_j\rangle\zeta_j, \zeta_k\rangle$. Here the inside $\langle\cdot\rangle$ is the $\mathcal{E}$-valued inner product on Z. A similar formula gives the matrix norms (see [10] Lemma 2.13). The restriction map $_{\mathcal{E}}CB^{ess}(W, H) \to {}_{\mathcal{A}}CB^{ess}(X, H)$ may thus be rewritten as the map $R : Z \otimes_{h\mathcal{E}} H \to {}_{\mathcal{A}}CB^{ess}(X, H)$, given by $R(z \otimes \zeta)(x) = (x, z)\zeta$. By 3.8, we need to check that R is a complete isometry (R is onto by similar considerations to those in the proof of the previous theorem). If $_{\mathcal{A}}CB^{ess}(X, H)$ is known to be a Hilbert column space (which is the case, say, if we know that X is a left $\mathcal{A}$-rigged module [3]), then we need only check that R is an isometry, or equivalently that

$$\sum_{k,j}\langle\langle z_k|z_j\rangle\zeta_j, \zeta_k\rangle \;\leq\; \sup\{\|[\sum_k (x_{ij}, z_k)\zeta_k]\|^2\} \;,$$

whenever $z_1, \cdots, z_n \in Z$, $\zeta_1, \cdots, \zeta_n \in H$, where the supremum is taken over all sized matrices $[x_{ij}]$ of norm 1 with entries in X.

The second part of 3.8 gives another condition which is equivalent to the above, and which may be easier to check in a concrete example: namely that the canonical completely contractive map $S : H_r \otimes_{h\mathcal{A}} X \to H_r \otimes_{h\mathcal{E}} W$, is an isometry. In this case $H_r \otimes_{h\mathcal{E}} W$ is a row Hilbert space, so that if S is an isometry then it is automatically a 'row-isometry'.

By symmetry, the subcontext $(\mathcal{A}, \mathcal{B}, X, Y)$ of $(\mathcal{E}, \mathcal{F}, W, Z)$ is right dilatable if and only if $X \otimes_{h\mathcal{B}} K \cong W \otimes_{h\mathcal{F}} K$ and $Y \otimes_{h\mathcal{A}} H \cong Z \otimes_{h\mathcal{E}} H$ isometrically, via the canonical maps, for all Hilbert $\mathcal{E}$-modules H and all Hilbert $\mathcal{F}$-modules K. Clearly, the condition $Y \otimes_{h\mathcal{A}} H \cong Z \otimes_{h\mathcal{E}} H$ for example, is equivalent to saying that the $\otimes_{h\mathcal{A}}$-norm on $Y \otimes H$ equals:

$$\|\sum_{j=1}^n y_j \otimes \zeta_j\|^2 = \sum_{k,j}\langle\langle y_k|y_j\rangle\zeta_j, \zeta_k\rangle \;, \tag{$\dagger$}$$

for $y_1, \cdots, y_n \in Y$, $\zeta_1, \cdots, \zeta_n \in H$. Thus 'right dilatability' is equivalent to saying that the induced functors $F_Y = Y \otimes_{h\mathcal{A}} -$ and $G_X = X \otimes_{h\mathcal{B}} -$ coincide, on the categories of Hilbert $\mathcal{E}-$ and $\mathcal{F}-$modules, with the weak Morita equivalence induced by Z and W of these categories. Summarizing:

Corollary 5.8. *A subcontext $(\mathcal{A}, \mathcal{B}, X, Y)$ of a strong Morita equivalence of $\mathcal{E}$ and $\mathcal{F}$, is right dilatable if and only if the induced functors F_Y and G_X give back the original weak Morita equivalence as explained above. This is equivalent to ($\dagger$) holding for all Hilbert $\mathcal{E}$-modules H, and the analogous formula for $X \otimes_{h\mathcal{B}} K$ holding for all Hilbert $\mathcal{F}$-modules K. This is also equivalent to the canonical maps $X \otimes_{h\mathcal{B}} K \to W \otimes_{h\mathcal{F}} K$ and $Y \otimes_{h\mathcal{A}} H \to Z \otimes_{h\mathcal{E}} H$ being row isometric, where H and K are the universal representations of $\mathcal{E}$ and $\mathcal{F}$ respectively.*

<u>Proof.</u>   Only the last statement still needs a word of proof, and this is similar to the proof that (iv) implies (iii) in 4.2.   □

A simple modification of the first of our examples of subcontexts shows again that the dense range condition in the definition of a subcontext is necessary for the corollary to hold. Without it one may have F_Y and F_X giving the same weak Morita equivalence between $_\mathcal{E}HMOD$ and $_\mathcal{F}HMOD$ as F_Z and F_W, without $\mathcal{A}$ and $\mathcal{B}$ being weakly Morita equivalent.

This ends our discussion of subcontexts. A natural question is if there is a comparable theory of quotient Morita contexts: We end this section with a simple but important observation, which for some reason we overlooked when writing [10]. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent operator algebras, and that $(\mathcal{A}, \mathcal{B}, X, Y)$ is the associated Morita context. Suppose that $\pi : \mathcal{A} \to \mathcal{D}$ is a completely contractive homomorphism into an operator algebra $\mathcal{D}$. Then if $\mathcal{E}$ is the closure of the range of π, then there exists a natural Morita context $(\mathcal{E}, \mathcal{F}, P, Q)$, which we shall call the *pushout* of $(\mathcal{A}, \mathcal{B}, X, Y)$ along π, which one may construct as follows. Suppose that $\mathcal{E}$ is a nondegenerate subalgebra of $B(H)$. Then π may be viewed as a representation of $\mathcal{A}$ on H. The original Morita context gives rise to a Hilbert space $K = Y \otimes_{h\mathcal{A}} H$, as in [10] Theorem 3.10, and an induced representation θ of $\mathcal{B}$ on K. Indeed, since $\mathcal{A}$ is strongly Morita equivalent to its linking algebra, we obtain an induced completely contractive representation ρ of the linking algebra $\mathcal{L}$ of the Morita context $(\mathcal{A}, \mathcal{B}, X, Y)$ (see §5 of [10]) on a Hilbert space $N = S \otimes_{h\mathcal{A}} H$, where S is the bimodule for the equivalence of $\mathcal{A}$ and its linking algebra. In fact $S = \mathcal{A} \oplus_c Y$, with notation as in [3]. By the associativity relations on p. 411 of that paper, we see that $N = H \oplus K$. Indeed we have recaptured the 'obvious' representation of $\mathcal{L}$ on $H \oplus K$, namely the one which is determined by

$$\rho\left(\left[\begin{array}{cc} a & x \\ y & b \end{array}\right]\right)\left[\begin{array}{c} \zeta \\ y' \otimes \xi \end{array}\right] = \left[\begin{array}{c} \pi(a)\zeta + \pi((x, y'))\xi \\ y \otimes \zeta + by' \otimes \xi \end{array}\right] .$$

Here $\zeta, \xi \in H$. The image under ρ inside $B(H \oplus K)$, of the four corners of the linking algebra $\mathcal{L}$, gives a Morita context implementing a strong Morita equivalence of $\mathcal{E}$ and the operator algebra which is the closure of $\theta(\mathcal{B})$. This is because the complete quotient conditions in the definition of strong Morita equivalence (3.1 of [10]), may be checked by the lifting criterion of 2.11 of [10]. This criterion, loosely speaking, is in terms of writing the c.a.i. of the algebras in terms of elementary tensors of Haagerup norm < 1. However, if $\{e_\alpha\}$ is a c.a.i. for $\mathcal{A}$, then $\pi(e_\alpha)$ is a c.a.i. for $\mathcal{E}$; and the associated aforementioned elementary tensors in $X \otimes Y$, are taken, via the completely contractive ρ, to elementary tensors in the new context, of Haagerup norm < 1. Similarly for a c.a.i. for $\mathcal{B}$.

A special case of the pushout occurs when π is the quotient homomorphism associated with a closed 2-sided ideal. This case was been studied recently, independently, and in much greater detail, in [30]. For C*−algebras of course this is not a 'special case', but an equivalent formulation, and was worked out in [40]. Note that a pushout or 'quotient Morita context' of a C*−context is again a C*−context, because completely contractive homomorphisms on C*−algebras are *-homomorphisms. On the other hand, it might be interesting to determine when the pushout of a strong Morita equivalence of operator algebras is a C*−context.

6. The linking algebras of an operator space or operator bimodule.

We turn to another interesting connection between the maximal C*−algebra of an operator algebra, and Morita equivalence/induced representations. It is also interesting in that it gives rise to a class of examples of Hilbert modules and C*-modules which may be associated to any operator space or operator module. It may also be viewed as a generalization of Example 2.4, along an avenue opened up by C. Zhang in [44]. He however was studying different questions, and was interested in the C*−envelope. For clarity we will give the idea first in the operator space case, and then later discuss the more general operator bimodule case.

Let X be any operator space. Assume that $X \subset B(K, H)$ completely isometrically. We form an operator system $\mathcal{S}$ consisting of matrices

$$\left[\begin{array}{cc} \lambda_1 I_H & x \\ y^* & \lambda_2 I_K \end{array} \right]$$

where $x, y \in X, \lambda_1, \lambda_2 \in \mathbb{C}$. In this section X^* will always mean the space of adjoints, not the dual space. A simple modification of Lemma 7.1 of [32], or Theorem 6.1 below, shows that $\mathcal{S}$ is independent of the particular H, K chosen, up to completely isometric isomorphism. Setting the 2-1 corner equal to 0, gives a unital operator algebra $\mathcal{U}(X)$, which only depends on the operator space structure of X. Let $\mathcal{U}_d(X)$ be the subalgebra with repetition on the diagonal, and let $\mathcal{A} = \mathcal{U}(X), \mathcal{A}_d = \mathcal{U}_d(X)$. Let $\mathcal{L}(X) = C^*_{max}(\mathcal{A})$ and $\mathcal{L}_d(X) = C^*_{max}(\mathcal{A}_d)$. Given a completely contractive unital representation π of $\mathcal{A}_d$ on a Hilbert space N, the restriction of π to the $1-2$ corner gives a completely contractive linear map $\phi : X \to B(N)$. Since $[\phi(X)N]$ and $(\cap_{x \in X} \ker \phi(x))^\perp$ are nontrivial complementary subspaces of N, we obtain a a nontrivial decomposition $N = H \oplus K$ say, with respect to which ϕ may be viewed as a map $X \to B(K, H)$. Using the aforementioned modification of the result in [32], ϕ may be 'extended' to a contractive unital representation $\tilde{\pi}$ of $\mathcal{A}$ on N, which is also an extension of π. It follows from this that $\mathcal{L}_d(X)$ is a unital C$^*-$subalgebra of $\mathcal{L}(X)$.

Thus there are 1-1 correspondences between the following classes: 1) completely contractive linear maps $X \to B(K, H)$, 2) unital completely contractive representations of $\mathcal{U}(X)$ on a Hilbert space $N(= H \oplus K)$, and 3) unital *-representations of $\mathcal{L}(X)$ on $N(= H \oplus K)$; and moreover one may use $\mathcal{U}_d(X)$ and $\mathcal{L}_d(X)$ instead of $\mathcal{U}(X)$ and $\mathcal{L}(X)$ in 2) and 3). If X is a maximal operator space, then one may remove the words 'completely' in 1) and 2) above.

The canonical projections e_1, e_2 in $\mathcal{A}$ give a decomposition of $\mathcal{L}(X)$ and $\mathcal{L}_d(X)$ as 2×2 matrices. By computations similar to the exercise 2.4, which are done more explicitly in [44], one sees that $e_1 \mathcal{L}(X) e_1$ is the closed linear span of e_1 and terms of the form $(xy^*)^n$, where $n \in \mathbb{N}$ and $x, y \in X$, and the products here are with respect to $\mathcal{L}(X)$. Write $\mathcal{C}$ or $C^*_{max}(XX^*)$ for $e_1 \mathcal{L}(X) e_1$, write $\mathcal{D}$ or $C^*_{max}(X^*X)$ for $e_2 \mathcal{L}(X) e_2$, and write W for the '1-2 corner' $e_1 \mathcal{L}(X) e_2$. We call W the *maximal C$^*-$correspondence of X*, for reasons which will be apparent later. Clearly $\mathcal{L}(X)$ may be rewritten as the closure of:

$$\left[\begin{array}{cc} \mathcal{C} & \mathcal{C}X \\ X^*\mathcal{C} & \mathcal{D} \end{array} \right] \ .$$

Example 2.4 shows that $C^*_{max}(XX^*) \cong C^*_{max}(XX^*) \cong C([0,1])$ and $W = C_0((0,1])$ if X is a one dimensional operator space.

Let H, K be general Hilbert spaces. Note that by the aforementioned modification of the Lemma 7.1 from [32], every completely contractive linear map $T : X \to B(K, H)$ gives a completely contractive unital representation of $\mathcal{A}$ on $H \oplus K$, and hence a *-representation of $\mathcal{L}(X)$ on $H \oplus K$, and by restriction, a unital *-representation π_T of $C^*_{max}(X^*X)$ on K. Notice that $\pi_T(x^*y) = T(x)^*T(y)$, for all $x, y \in X$, and clearly there can be only one such unital *-representation of $C^*_{max}(X^*X)$ with this property. We shall call this construction of π_T from T the universal property of $C^*_{max}(X^*X)$. One has a similar universal property for $C^*_{max}(XX^*)$.

The converse is also true, namely, that any unital *-representation π of $C^*_{max}(X^*X)$ on a Hilbert space K gives rise to a completely contractive linear map $S_\pi : X \to B(K, H_\pi)$, for some Hilbert space H_π. Indeed let $M = X \otimes K$, and define a semi-inner-product on M by $\langle x_1 \otimes \eta_1, x_2 \otimes \eta_2 \rangle = \langle \pi(x_2^*x_1)\eta_1, \eta_2 \rangle$. We define H_π or $X \otimes_\pi H$ to be the completion of the quotient of M by the null vectors in M. Then H_π is a Hilbert space, and we define $S_\pi(x)(\eta) = x \otimes \eta$, for $x \in X, \eta \in K$.

It is easily checked that S_π is completely contractive, and also that $\pi(x^*y) = S_\pi(x)^* S_\pi(y)$, for all $x, y \in X$. Thus $\pi_{S_\pi} = \pi$. We also note that $[S_\pi(X)(K)]$ is dense in H_π.

Finally note that if one begins with a completely contractive linear $T : X \to B(K, H)$, and then forms the associated representation $\pi = \pi_T$ of $C^*_{max}(X^*X)$ on K as above, and then produces a Hilbert space H_π and complete contraction $S = S_{\pi_T}$ as in the last paragraph. Then it is clear that there is a canonical (and indeed unique) isometry $U : H_\pi \to H$ with the property that $US = T$. We note that U is a unitary if and only if the span of $T(X)(K)$ is dense in H. The above seems to be some kind of 'polar decomposition' for operator spaces. We have written a general T as the composition of an isometry, and a map S of the 'standard form' $S(x)(\eta) = x \otimes \eta$.

If one begins with a *-representation θ of $C^*_{max}(XX^*)$ on H, one defines $K_\theta = X^* \otimes_\theta H$ similarly, and we define a map $R_\theta : X \to B(K_\theta, H)$ given by $R_\theta(x)(y^* \otimes \zeta) = \theta(xy^*)(\zeta)$. Now $R_\theta(x)R_\theta(y)^* = \theta(xy^*)$. If θ comes from a map $T : X \to B(K, H)$, via the universal property of $C^*_{max}(XX^*)$, then one can easily check that there is a coisometry $V : K \to K_\theta$ such that $T = R_\theta V$. Also, V is unitary if and only if $\cap_{x \in X} \ker T(x) = (0)$.

The universal property of $C^*_{max}(X^*X)$ is reminiscent of the property of the universal C^*−algebra $C^*(X)$ of an operator space X [36]. However, $C^*(X) \cong C^*_{max}(OA(X))$, where $OA(X)$ is the universal operator algebra of an operator space discussed in [37]. Indeed there is no obvious inclusion of X in $C^*_{max}(X^*X)$. It is also not true that $\mathcal{L}(X)$ coincides with the universal C^*−algebra generated by the operator system $\mathcal{S}$. To see this, observe that the latter C^*−algebra is shown in [21] to be nonexact if $\mathcal{S} = M_2$, whereas in this case $X = \mathbb{C}$ and $\mathcal{L}(X)$ is the (nuclear) C^*−algebra in Example 2.4. However it is clear that $\mathcal{L}(X)$ is always a quotient C^*−algebra of the C^*−algebra of the operator system $\mathcal{S}$.

It would be interesting to study these universal C^*−algebras for some of the common finite dimensional operator spaces X. Understanding $C^*_{max}(X^*X)$ for $X = \ell^1_n$, for example, corresponds to understanding a certain von Neumann type inequality. One must find n universal contractions which together with their adjoints, satisfy certain polynomial inequalities. As far as we know, this particular type of von Neumann type inequality, or such universal C^*−algebras, have not been studied.

As noted in [44], the subalgebra $C^*_0(XX^*)$ of $C^*_{max}(XX^*)$ generated by XX^* (but not e_1), is, by construction, strongly Morita equivalent to $C^*_0(X^*X)$. Thus it is not strange that these C^*−algebras have the same 'representation theory'. We can rephrase the construction of S_π from π given above as follows. A unital *-representation π of $C^*_{max}(X^*X)$ on a Hilbert space K, restricts to a *-representation of $C^*_0(X^*X)$. By the basic theory of strong Morita equivalence, π gives rise to a canonical second Hilbert space H (which may be obtained as the 'interior' or 'module Haagerup' tensor product of W and K (see [22, 4] for example)). We also obtain a canonical *-representation of $C^*_{max}(X^*X)$ on H, and a canonical *-representation of $\mathcal{L}(X)$ on the Hilbert space $H \oplus K$. This is the whole point of 'induced representations' [39]. By restriction to the 1-2 corner, we obtain the canonical completely contractive linear map $S_\pi : X \to B(K, H)$.

Looking at $C^*_{max}(X^*X)$ from the point of view of C^*−modules is perhaps the best way to formulate its universal property. Namely, if one takes $W = e_1\mathcal{L}(X)e_2$, then as we just saw, W is a right C^*−module over $C^*_{max}(X^*X)$, and there is an obvious complete isometry $i : X \to W$ such that the identity and the range of $i(X)^*i(X)$ generates $C^*_{max}(X^*X)$. Moreover, any completely contractive map $T : X \to Z$ into a right C^*−module Z over $\mathcal{B}$, say, give rise to a (necessarily unique) unital *-homomorphism $\pi : C^*_{max}(X^*X) \to \mathcal{B}$ such that $\pi(i(y)^*i(x)) = \langle T(y)|T(x) \rangle$ for all $x, y \in X$. This may be seen by applying the previous universal property of $C^*_{max}(X^*X)$ to T, the latter viewed as a map into the range of a concrete faithful *-representation of the linking C^*−algebra of Z. Conversely any unital *-homomorphism π from $C^*_{max}(X^*X)$ into a C^*-algebra

$\mathcal{B}$, restricts to a *-homomorphism π' on $C_0^*(X^*X)$, which induces a quotient Morita context for the range of π' as in [40] §3. This is the 'pushout' construction discussed at the end of our last section. We obtain a C$^*-$bimodule Z and a complete contraction $T : X \to Z$ with the $\pi(i(y)^*i(x)) = \langle T(y)|T(x)\rangle$ property.

We now generalize the above to give the linking algebras, and the 'maximal C$^*-$correspondence' of an operator bimodule. Let X be an $\mathcal{A} - \mathcal{B}$-operator module, and suppose that $\mathcal{C}$ and $\mathcal{D}$ are the maximal C$^*-$algebras of $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\tilde{X}$ be the $\mathcal{C} - \mathcal{D}$-dilation of X, namely $\tilde{X} = \mathcal{C} \otimes_{h\mathcal{A}} X \otimes_{h\mathcal{B}} \mathcal{D}$. As we mentioned at the end of §3, this 'bi-dilation' has the obvious universal property. For simplicity in what follows we shall assume $\mathcal{A}$ and $\mathcal{B}$ are unital. The more general case of c.a.i.'s follows from the unital case by the usual tricks. By the Christensen-Effros-Sinclair representation theorem for operator bimodules over C$^*-$algebras, there is a concrete representation of $\mathcal{C}, \mathcal{D}$, and $\tilde{X}$ on Hilbert spaces H' and K' say, in such a way that the module actions become concrete multiplication of operators. It can also be arranged, by direct summing with the identity representations of the C$^*-$algebras, that the representations of $\mathcal{C}$ and $\mathcal{D}$ on H' and K' respectively in the Christensen-Effros-Sinclair representation are faithful (and hence completely isometric). We shall refer to this as a *faithful CES representation*. With respect to one fixed faithful CES representation we form an operator system $\mathcal{S}$ in $B(H' \oplus K')$ consisting of matrices

$$\begin{bmatrix} c & \tilde{x} \\ \tilde{y}^* & d \end{bmatrix}$$

where $c \in \mathcal{C}, d \in \mathcal{D}, \tilde{x}, \tilde{y} \in \tilde{X}$. Paulsen's 2×2 matrix tricks, used in the proof of Lemma 7.1 in [32], or theorem 2.4 in [41], for example may be used in an analogous and straightforward fashion to yield the following:

Theorem 6.1. *Given a *-representation θ of $\mathcal{C}$ on a Hilbert space H and a *-representation π of $\mathcal{D}$ on a Hilbert space K, and given a completely contractive $\mathcal{C} - \mathcal{D}$-module map $\Phi : \tilde{X} \to B(K, H)$, then the map Ψ from $\mathcal{S}$ into $B(H \oplus K)$ defined by*

$$\begin{bmatrix} c & \tilde{x} \\ \tilde{y}^* & d \end{bmatrix} \mapsto \begin{bmatrix} \theta(c) & \Phi(\tilde{x}) \\ \Phi(\tilde{y})^* & \pi(d) \end{bmatrix}$$

is completely positive.

We will omit the proof. From this one can immediately deduce that the operator system structure on $\mathcal{S}$ is independent of the particular faithful CES representation of $\tilde{X}$. Note that this theorem, seems to give a direct proof of 4.1, using the idea in [41] of extending the c.p. map Ψ and then using 4.2 in [42] to prove that the 1-2 corner of the extension is still a bimodule map. We have not checked the details, since it is clearly more trouble than the proof we gave, and moreover still requires fussing with the non-nondegeneracy of the representations involved.

We define the *upper triangular operator algebra* or *upper linking operator algebra*, $\mathcal{U}(X)$ to be the set of matrices

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$$

in $\mathcal{S}$, where $a \in \mathcal{A}, b \in \mathcal{B}, x \in X$. We define $\mathcal{U}(\tilde{X})$ similarly, except that the entries come from $\mathcal{C}, \mathcal{D}$ and $\tilde{X}$. By looking at the concrete realization of $\mathcal{S}$ that we began with, it is easily seen that with respect to the natural multiplication on $B(H' \oplus K')$, the spaces $\mathcal{U}(X)$ and $\mathcal{U}(\tilde{X})$ are operator algebras. Note that the adjoint in $\mathcal{S}$ of the copy of X in the 1-2 corner, is $\bar{X}$, the conjugate operator module of X mentioned at the end of §1. We now form $\mathcal{L}(X) = C_{max}^*(\mathcal{U}(X))$, and call this the

linking C$^*-$algebra of X. We let $\mathcal{L}(\tilde{X})$ be $C^*_{max}(\mathcal{U}(\tilde{X}))$. It is easily seen that $\mathcal{L}(X)$ has a natural decomposition as 2×2 matrices, and we define $\mathcal{E}$ to be the 1-1 corner $e_1\mathcal{L}(X)e_1$, W to be the 1-2 corner $e_1\mathcal{L}(X)e_2$, and $\mathcal{F} = e_2\mathcal{L}(X)e_2$. Here e_1 and e_2 are the copies of the identities of $\mathcal{A}$ and $\mathcal{B}$.

If one begins with a completely contractive $\mathcal{A} - \mathcal{B}$-bimodule map $T : X \to B(K, H)$, where H and K are $\mathcal{A}$- and $\mathcal{B}$-Hilbert modules respectively, then by the universal property of the dilation $\tilde{X}$, there is a unique completely contractive $\mathcal{C} - \mathcal{D}$-bimodule extension $\tilde{T} : \tilde{X} \to B(K, H)$. By 6.1, we obtain a completely positive unital $\Psi : \mathcal{S} \to B(H \oplus K)$, and by restriction, a completely contractive unital homomorphism σ on $\mathcal{U}(\tilde{X})$, and another, σ', on $\mathcal{U}(X)$. Conversely any completely contractive unital homomorphism $\sigma' : \mathcal{U}(X) \to B(N)$ determines a decomposition $N = H \oplus K$, and *-representations π and θ on K and H respectively, and a completely contractive $\mathcal{A} - \mathcal{B}$-bimodule map $X \to B(K, H)$. Thus, as before, there are 1-1 correspondences between the four classes of maps whose elements we have labeled above with symbols $T, \tilde{T}, \sigma$, and σ'.

By the universal property of the maximal C$^*-$algebra, σ extends to a *-representation $\tilde{\sigma}$ of $\mathcal{L}(\tilde{X})$. The restriction of $\tilde{\sigma}$ to $\mathcal{U}(X)$ clearly coincides with σ', which shows that $\mathcal{L}(X)$ may be taken to be the C$^*-$algebra inside $\mathcal{L}(\tilde{X})$ generated by $\mathcal{U}(X)$. Hence $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$. Clearly, one sees also that $\mathcal{S}$ sits naturally inside $\mathcal{L}(X)$. Thus inside $\mathcal{L}(X)$ the product $[\mathcal{C}X\mathcal{D}]$ is completely isometrically isomorphic to $\mathcal{C} \otimes_{h\mathcal{A}} X \otimes_{h\mathcal{B}} D$. And we can add to our list of 1-1 correspondences between classes of maps, the correspondence between completely contractive $\mathcal{A} - \mathcal{B}$-bimodule maps $T : X \to B(K, H)$, and unital *-representations of $\mathcal{L}(X)$ on $N(= H \oplus K)$.

Notice the above gives a simple way of writing any completely contractive $\mathcal{A} - \mathcal{B}$-bimodule map $T : X \to B(K, H)$ as $P_H\pi(\cdot)|_K$ where π is a *-representation on $H \oplus K$ of a C$^*-$algebra which contains X. Using the CES representation theorem, one can replace the $B(K, H)$ in the last sentence by any $\mathcal{A} - \mathcal{B}$-operator bimodule V, to represent an $\mathcal{A} - \mathcal{B}$-bimodule map $T : X \to V$ as $R\pi(\cdot)V$ where R, V are isometric or coisometric module maps.

One may proceed as before, to show that there are 1-1 correspondences between completely contractive $\mathcal{A} - \mathcal{B}$-bimodule maps $T : X \to B(K, H)$, and *-representations π and θ of $\mathcal{F}$ and $\mathcal{E}$ on H and K respectively. This all goes through with no essential changes. Again it is interesting that one can write a general such T, as a standard form $S(x)(\eta) = x \otimes \eta$, multiplied by an isometry (or coisometry in the other case).

This universal property of $\mathcal{F}$ (and similarly the one for $\mathcal{E}$) is probably again best described in terms of C$^*-$modules. Again W is a right C$^*-$module over $\mathcal{F}$, and there is an obvious $\mathcal{A} - \mathcal{B}$-module map $i : X \to W$. Recall that a right $\mathcal{C} - \mathcal{G}$-C$^*-$correspondence V (also known as a $\mathcal{G}$-rigged $\mathcal{C}$-module) is a right C$^*-$module over a C$^*-$algebra $\mathcal{G}$, which is also a nondegenerate left Banach $\mathcal{C}-$module (see [4] §4). We define a *rigging map* to be a completely contractive left $\mathcal{A}$-module map $T : X \to V$ into a right $\mathcal{C} - \mathcal{G}$-correspondence, for which there exists a unital completely contractive homomorphism $\pi : \mathcal{B} \to \mathcal{G}$ such that $T(xb) = T(x)\pi(b)$ for all $b \in \mathcal{B}$. Notice that this makes T an $\mathcal{A} - \mathcal{B}$-bimodule map. Also note that the W above is a $\mathcal{C} - \mathcal{D}-C^*-$correspondence, and that the $i : X \to W$ is a rigging map. As before, we have a pushout construction. The relation $\langle T(x)|T(y)\rangle = \pi(i(x)^*i(y))$ for all $x, y \in X$, determines a correspondence between 1) rigging maps $T : X \to V$ into a right $\mathcal{C} - \mathcal{G}$-correspondence V, and 2) unital *-homomorphisms π' from $\mathcal{F}$ into a C*-algebra. We omit the details, and the standard adaption to the nonunital case. The other universal properties of $\mathcal{U}(X), \mathcal{L}(X)$ may also be stated in terms of C$^*-$modules and C$^*-$correspondences, but we will not take the time to do that here.

Final remark. We feel that there is some aspect missing in our understanding of operator modules. The fact that the notion we called C$^*-$restrictability in [7] is automatic suggests strongly the need for a good test for an $\mathcal{A}$-operator module to be a $\mathcal{C}$-operator module.

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